Lie Groupoids in Classical Field Theory I: Noether's Theorem

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Abstract

In the two papers of this series, we initiate the development of a new approach to implementing the concept of symmetry in classical field theory, based on replacing Lie groups/algebras by Lie groupoids/algebroids, which are the appropriate mathematical tools to describe local symmetries when gauge transformations are combined with space-time transformations. Here, we outline the basis of the program and, as a first step, show how to (re)formulate Noether's theorem about the connection between symmetries and conservation laws in this approach.

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1 Introduction

Symmetry is a fundamental concept of science deeply rooted in human culture, as can be verified by contemplating, e.g., the monumental collection of papers assembled in [9, 10], testifying the innumerous ways in which it permeates all areas of knowledge. In mathematics, it has been the driving force for the development of group theory, which began in the 19^{th} century with the work of E. Galois and S. Lie, formalizing the idea that symmetry transformations can be assembled into groups. And in physics, it has in the course of the 20^{th} century become one of the most influential guiding principles for the development of new theories, used in a wide variety of contexts, and now plays an important role in practically all areas, including mechanics and field theory, both classical and quantum.

Correspondingly, and not surprisingly, symmetries in physics appear in different variants. For example, they may act through transformations in 3-dimensional physical space such as translations, rotations and reflections (spatial symmetries), or in the case of relativistic physics where space and time merge into a single space-time continuum, transformations in 4-dimensional space-time such as Lorentz transformations (space-time symmetries), or else they may act only through transformations in an abstract internal space that has nothing to do with physical space or space-time, being instead related to the dynamical variables of the theory under consideration (internal symmetries). Similarly, they can be distinguished according to whether the corresponding group of transformations is continuous, such as the group of spatial translations or rotations (parametrized by vectors or by the Euler angles, say), or is discrete, such as the group of spatial translations or rotations in a crystal lattice or a reflection group. And finally, a very important extension of the usual symmetry concept arises from the idea that the parameters characterizing a specific element within the pertinent group may depend on the point in space or space-time where the symmetry transformation is performed. This notion of "gauging a symmetry", thus extending it from a global symmetry to a local symmetry by allowing different transformations to be performed at different points, is at the heart of gauge theories, which occupy a central position in modern field theory, classical as well as quantum. (We think of this as an extension of the symmetry concept because gauge transformations do not really represent symmetry transformations in the strict sense of the word, since they do not relate observable quantities. Instead, their presence reflects the fact that the system under consideration is being described in terms of redundant variables which are not observable, such as the potentials in electrodynamics, and the amount of redundancy in the choice of these variables is controlled by the principle of gauge invariance: the observable, physical content of the theory is encoded in its gauge invariant part.)

The mathematical implementation of different types of symmetries requires different types of groups. In particular, continuous symmetries are described in terms of Lie groups and, infinitesimally, of Lie algebras, and local symmetries involve infinite-dimensional Lie groups and Lie algebras. But more exotic mathematical entities have also begun to appear on the scene, essentially since the 1970's, and have come to be associated with generalized notions of symmetry, such as Lie superalgebras and quantum groups.

In this paper, we propose to employ yet another mathematical tool that is particularly well adapted to the concept of a local symmetry, namely that of Lie groupoids and, infinitesimally, of Lie algebroids. The main advantage of such an approach is that it eliminates the need for working from the very beginning with infinite-dimensional objects such as diffeomorphism

groups of manifolds and automorphism groups of bundles, whose mathematical structure is notoriously difficult to handle: such groups do arise along the way but are now derived from a more fundamental underlying entity, which is purely finite-dimensional.

This procedure of "reduction to finite dimensions" is by no means new and has in fact been used in differential geometry for a long time, namely when one considers a representation of a Lie group G by diffeomorphisms of a manifold M not as a group homomorphism $G \longrightarrow \mathrm{Diff}(M)$, whose continuity (let alone smoothness) is hard to define and even harder to control, but rather as a group action $G \times M \longrightarrow M$, for which the definition of continuity (and smoothness) is the obvious one. A similar procedure can be applied when G is replaced by one of the aforementioned infinite-dimensional groups which appear naturally in differential geometry and in gauge theories, the main difference being that in this case the corresponding action is no longer that of a Lie group on a manifold but rather that of a Lie groupoid on a fiber bundle.

The history of the theory of groupoids is to a certain extent parallel to that of group theory itself. The notion of an abstract groupoid goes back to a paper by H. Brandt in 1927 [3]; it applies to discrete as well as to continuous symmetries. Lie groupoids seem to have been introduced by C. Ehresmann in the 1950's, in conjunction with principal bundles and connections, but in contrast to these did not find their way into mainstream differential geometry for several decades. Lie algebroids and their relation to Lie groupoids were apparently first discussed by J. Pradines in 1968 [12], but the main result (Lie's third theorem) as stated there is incorrect and was only rectified much later [5]. At the time of this writing, the standard textbook in the area is Ref. [11], whose results we shall use freely in this paper.

We conclude this introduction with an outline of the contents. In Section 2, we briefly review the modern formulation of classical field theory in a geometric framework, where fields are sections of fiber bundles over space-time and hence coordinate invariance and gauge invariance (in the sense of invariance under changes of local trivializations of these fiber bundles) are built in from the very beginning. As in the standard geometric formulation of classical mechanics [1, 2], we shall for the sake of simplicity restrict ourselves to a first order formalism – lagrangian as well as hamiltonian. In Section 3, we discuss a few basic topics from the theory of Lie groupoids and Lie algebroids. In particular, we present the construction of the jet groupoid of a Lie groupoid, which is badly neglected in the mathematical literature (for instance, it does not appear at all in Ref. [11]) but which turns out to be of crucial importance for the entire theory; it can be iterated to produce a construction of the second order jet groupoid of a Lie groupoid and of a non-holonomous extension thereof which is needed in the sequel. Apart from that, our goal in these two introductory sections is essentially just to delineate the concepts we shall be using and to fix the notation. In Section 4, we discuss how an action of a Lie groupoid on a fiber bundle induces actions of certain Lie groupoids derived from the original one on certain fiber bundles derived from the original one. Here, the central point is that apart from obvious functorial procedures such as defining the action of the jet groupoid on the jet bundle or the tangent groupoid on the tangent bundle (of the total space), we are also able to construct a new – and much less obvious – induced action of the jet groupoid on the tangent bundle (of the total space). This construction is perhaps the most important result of the paper because it allows us to give a precise mathematical definition of invariance, under the action of a Lie groupoid on some fiber bundle, of geometric structures on its total space, at least when these are defined by some kind of tensor field, thus finally overcoming one of the major obstacles in the theory that has for a long time jeopardized its relevance for applications.

The new feature as compared to group actions is that such an invariance refers not to the original Lie groupoid itself but rather to its jet groupoid, or some subgrupoid thereof. We also exhibit the relation with the corresponding representations of the appropriate groups of bisections and, as an application, show in which sense the multicanonical form θ and the multisymplectic form ω of the covariant hamiltonian formalism, as presented in Section 2, are invariant under the appropriate induced actions. Finally, in Section 5, we introduce the concept of momentum map and prove Noether's theorem in this setting.

In the second paper of this series, we shall specialize the general formalism developed below to what may be regarded as the most important general class of geometric field theories: gauge theories. There, all bundles that appear are derived from a given principal bundle over space-time, either as an associated bundle (whose sections are matter fields) or as the corresponding connection bundle (whose sections are connections, or in physics language, gauge potentials), and the Lie groupoid acting on them is the corresponding gauge groupoid. In this context, we shall then be able to discuss issues such as the significance of the procedure of "gauging a symmetry", already mentioned above, the prescription of minimal coupling and Utiyama's theorem, generalizing the results of Ref. [6] from the context of Lie group bundles, which are sufficient to handle internal symmetries, to Lie groupoids, as required to deal with the general case of space-time symmetries mixed with internal symmetries.

2 Geometric formulation of classical field theory

We start out by fixing a fiber bundle E over a base manifold M, with bundle projection denoted by $\pi_E : E \longrightarrow M$: it will be called the *configuration bundle* since its sections are the basic fields of the theory under consideration. This requires that, in physics language, M is to be interpreted as space-time (even though we do not assume it to carry any fixed metric, given that in general relativity the metric tensor is itself a dynamical variable and hence cannot be fixed "a priori").

In order to formulate the laws governing the dynamics of the fields, we need to consider derivatives (velocities), as well as their duals (momenta).

In order to do so, we begin by introducing the *jet bundle* JE of E, together with the *linearized jet bundle* $\vec{J}E$ of E, as follows: for any point e in E with base point $x = \pi_E(e)$ in M, let $L(T_xM, T_eE)$ denote the space of linear maps from the tangent space T_xM to the tangent space T_eE and consider the affine subspace

$$J_eE \;=\; \{\, u_e \in L(T_xM, T_eE) \mid T_x\pi_E \circ u_e = \mathrm{id}_{T_xM} \,\} \tag{1}$$

and its difference vector space

$$\vec{J}_eE \;=\; \{\, \vec{u}_e \in L(T_xM, T_eE) \mid T_x\pi_E \circ \vec{u}_e = 0 \,\} \tag{2}$$

i.e.,

$$\vec{J}_eE \;=\; L(T_xM, V_eE) \;=\; T_x^*M \otimes V_eE \tag{3}$$

where $V_eE = \ker T_e\pi_E$ is the vertical space of E at e. Taking the disjoint union as e varies over E, this defines JE and $\vec{J}E$ as bundles in two different ways, which will collectively be

referred to as *jet bundles*: over E, JE is an affine bundle and $\vec{J}E$ is a vector bundle with respect to the corresponding jet target projections $\pi_{JE} : JE \longrightarrow E$ and $\pi_{\vec{J}E} : \vec{J}E \longrightarrow E$, while over M, both of them are fiber bundles with respect to the corresponding jet source projections (obtained from the former by composition with the original bundle projection π_E).[1] Moreover, composition with the appropriate tangent maps provides a canonical procedure for associating with every strict homomorphism $f : E \longrightarrow F$ of fiber bundles E and F over M a map $Jf : JE \longrightarrow JF$ (sometimes called its jet prolongation or jet extension), which is a homomorphism of affine bundles (i.e., a fiberwise affine smooth map) covering f, together with a map $\vec{J}f : \vec{J}E \longrightarrow \vec{J}F$, which is a homomorphism of vector bundles (i.e., a fiberwise linear smooth map) covering f; in particular, both are again strict homomorphisms of fiber bundles over M, so in fact J and $\vec{J}$ are *functors* in the category of fiber bundles over a fixed base manifold. This is summarized in the following commuting diagrams:

$$
\begin{tikzcd}
JE \arrow[rr, "Jf"] \arrow[d] & & JF \arrow[d] \\
E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
\vec{J}E \arrow[rr, "\vec{J}f"] \arrow[d] & & \vec{J}F \arrow[d] \\
E \arrow[rr, "f"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\
& M &
\end{tikzcd}
\tag{4}
$$

Explicitly, given $e \in E$ with $\pi_E(e) = x$, $u_e \in J_eE \subset L(T_xM, T_eE)$, $\vec{u}_e \in \vec{J}_eE = L(T_xM, V_eE)$, we have

$$
J_ef(u_e) \; = \; T_ef \circ u_e \quad , \quad \vec{J}_ef(\vec{u}_e) \; = \; T_ef \circ \vec{u}_e \; . \tag{5}
$$

In particular, for any section φ of E, we have

$$
j(f \circ \varphi) \; = \; Jf \circ j\varphi \; . \tag{6}
$$

The important role jet bundles play in differential geometry is largely due to the fact that they provide the adequate geometric setting for taking derivatives of sections. More precisely, any section of E, say φ, induces canonically a section of JE which is often called its *jet prolongation* or *jet extension* and which we may denote by $j\varphi$, as in much of the mathematical literature, or by $(\varphi, \partial\varphi)$, to indicate that it contains all the information about the values of φ and of its first order derivatives at each point of M. But this prolongation is really just a reinterpretation of the tangent map $T\varphi$ to φ, since φ being a section of E implies that, for any $x \in M$, $T_x\varphi \in J_{\varphi(x)}E \subset L(T_xM, V_{\varphi(x)}E)$. Obviously, given $e \in E$ with $\pi_E(e) = x$, every jet $u_e \in J_eE$ can be represented as the derivative at x of some section φ of E satisfying $\varphi(x) = e$, i.e., we can always find φ such that $u_e = T_x\varphi$, but this does of course not mean that every section of JE, as a fiber bundle over M, can be written as the jet prolongation of some section of E: those that can be so written are called *holonomous*, and it is then easy to see that a section $\tilde{\varphi}$ of JE will be holonomous if and only if $\tilde{\varphi} = j\varphi$ where $\varphi = \pi_{JE} \circ \tilde{\varphi}$.

In passing, we note that sections of JE not as a fiber bundle over M but as an affine bundle over E also have an important role to play: they correspond to *connections* in E, realized through

[1] Here and throughout this paper, we face the problem that the same expressions "source" and "target" are used in the theory of jets and in the theory of groupoids with different meanings. We shall avoid confusion by adhering to the convention to use the prefix "jet" in the first case.

their *horizontal lifting map* (of tangent vectors). And if we fix a connection $\Gamma : E \longrightarrow JE$, we can introduce the notion of *covariant derivative* of a section φ: this is then a section of $\vec{J}E$ which we may denote by $(\varphi, D\varphi)$ and which is defined as the difference $T\varphi - \Gamma \circ \varphi$. Summarizing, we may specify the statement at the beginning of the previous paragraph by saying that the jet bundle and the linearized jet bundle provide the adequate geometric setting for taking ordinary (partial) derivatives and for taking covariant derivatives of sections, respectively.

In order to handle second order derivatives, we shall need the *second order jet bundle* J^2E: this can be constructed either directly or else by iteration of the previous construction and subsequent reduction, which occurs in two steps. First, observe that the iterated first order jet bundle $J(JE)$ admits two natural projections to JE, namely, the standard projection $\pi_{J(JE)} : J(JE) \longrightarrow JE$ and the jet prolongation $J\pi_{JE} : J(JE) \longrightarrow JE$ of the standard projection $\pi_{JE} : JE \longrightarrow E$, explicitly defined as follows: for $e \in E$, $u_e \in J_eE$ and $u'_{u_e} \in J_{u_e}(JE)$,

$$(\pi_{J(JE)})_{u_e}(u'_{u_e}) = u_e \,, \tag{7}$$

whereas

$$(J\pi_{JE})_{u_e}(u'_{u_e}) = T_{u_e}\pi_{JE} \circ u'_{u_e} \,. \tag{8}$$

They fit into the following commutative diagram:

$$\begin{tikzcd}
 & J(JE) \arrow[dl, "\pi_{J(JE)}"'] \arrow[dr, "J\pi_{JE}"] & \\
JE \arrow[dr, "\pi_{JE}"'] & & JE \arrow[dl, "\pi_{JE}"] \\
 & E &
\end{tikzcd} \tag{9}$$

The first step of the reduction mentioned above is then to restrict to the subset of $J(JE)$ where the two projections coincide: for $e \in E$ and $u_e \in J_eE$, define

$$\bar{J}^2_{u_e}E \;=\; \{\, u'_{u_e} \in J_{u_e}(JE) \mid \pi_{J(JE)}(u'_{u_e}) = J\pi_{JE}(u'_{u_e}) \,\} \,. \tag{10}$$

Taking the disjoint union as u_e varies over JE, this defines what we shall call the *semiholonomic second order jet bundle* of E, denoted by $\bar{J}^2E$: it is naturally a fiber bundle over M and, since $\pi_{J(JE)}$ and $J\pi_{JE}$ are both homomorphisms of affine bundles, it is also an affine bundle over JE. The second step consists in decomposing this, as a fiber product of affine bundles over JE, into a symmetric part and an antisymmetric part: the former is precisely J^2E and is an affine bundle over JE, with difference vector bundle equal to the pull-back to JE of the vector bundle $\pi_E^*\big(\bigvee^2 T^*M\big) \otimes VE$ over E by the jet target projection π_{JE}, whereas the latter is a vector bundle over JE, namely the pull-back to JE of the vector bundle $\pi_E^*\big(\bigwedge^2 T^*M\big) \otimes VE$ over E by the jet target projection π_{JE}:

$$\begin{aligned} \bar{J}^2E \;&\cong\; J^2E \times_{JE} \pi_{JE}^*\Big(\pi_E^*\big(\textstyle\bigwedge^2 T^*M\big) \otimes VE\Big) \\ \vec{J}^2E \;&\cong\; \pi_{JE}^*\Big(\pi_E^*\big(\textstyle\bigvee^2 T^*M\big) \otimes VE\Big) \end{aligned} \tag{11}$$

But what is perhaps even more important is the following observation: given any section $\tilde{\varphi}$ of JE, its jet prolongation $j\tilde{\varphi}$ will be a section of $J(JE)$ which will take values in $\bar{J}^2E$ if and only

if $\tilde{\varphi}$ is holonomous. (Indeed, let us set $\varphi = \pi_{JE} \circ \tilde{\varphi}$. If $\tilde{\varphi}$ is holonomous, then as already observed above, we must have $\tilde{\varphi} = j\varphi$ and hence $J\pi_{JE} \circ j\tilde{\varphi} = j(\pi_{JE} \circ \tilde{\varphi}) = j\varphi = \tilde{\varphi} = \pi_{J(JE)} \circ j\tilde{\varphi}$, i.e., $j\tilde{\varphi} = j^2\varphi$ takes values in $\bar{J}^2E$. Conversely, if $j\tilde{\varphi}$ takes values in $\bar{J}^2E$, then $j\varphi = j(\pi_{JE} \circ \tilde{\varphi}) = J\pi_{JE} \circ j\tilde{\varphi} = \pi_{J(JE)} \circ j\tilde{\varphi} = \tilde{\varphi}$.) Of course, this means exactly that $j\tilde{\varphi} = j^2\varphi$ is a section of J^2E, so we may characterize J^2E as the unique "maximal holonomous subbundle" of $\bar{J}^2E$, i.e., that subbundle of $\bar{J}^2E$ whose sections are precisely the holonomous sections of $\bar{J}^2E$. For more details, see [13, Chapter 5].

Returning to the first order formalism, the next step consists in taking duals. Briefly, the *affine dual* $J^\star E$ of JE and the *linear dual* $\vec{J}^* E$ of $\vec{J}E$ are defined as follows: for any point e in E with base point $x = \pi_E(e)$ in M, put

$$J_e^\star E \;=\; \{\, z_e : J_eE \longrightarrow \mathbb{R} \mid z_e \text{ is affine} \,\} \tag{12}$$

and

$$\vec{J}_e^* E \;=\; \{\, \vec{z}_e : \vec{J}_eE \longrightarrow \mathbb{R} \mid \vec{z}_e \text{ is linear} \,\} \tag{13}$$

However, the multiphase spaces of field theory are defined with an additional twist, which consists in replacing the real line by the one-dimensional space of volume forms on the base manifold M at the appropriate point. In other words, the *twisted affine dual* $J^{\circledast}E$ of JE and the *twisted linear dual* $\vec{J}^{\circledast}E$ of $\vec{J}E$ are defined as follows: for any point e in E with base point $x = \pi_E(e)$ in M, put

$$J_e^{\circledast} E \;=\; \{\, z_e : J_eE \longrightarrow \textstyle\bigwedge^n T_x^*M \mid z_e \text{ is affine} \,\} \tag{14}$$

and

$$\vec{J}_e^{\circledast} E \;=\; \{\, \vec{z}_e : \vec{J}_eE \longrightarrow \textstyle\bigwedge^n T_x^*M \mid \vec{z}_e \text{ is linear} \,\} \tag{15}$$

Taking the disjoint union as e varies over E, this defines $J^\star E$, $\vec{J}^* E$, $J^{\circledast}E$ and $\vec{J}^{\circledast}E$ as bundles in two different ways, which will collectively be referred to as *cojet bundles*: all of them are vector bundles over E with respect to the corresponding cojet target projections and fiber bundles over M with respect to the corresponding cojet source projections (obtained from the former by composition with the original bundle projection π_E). Considered as vector bundles over E, we have

$$J^{\circledast}E \;=\; J^\star E \otimes \pi_E^*\big(\textstyle\bigwedge^n T^*M\big) \tag{16}$$

and

$$\vec{J}^{\circledast}E \;=\; \vec{J}^* E \otimes \pi_E^*\big(\textstyle\bigwedge^n T^*M\big) \tag{17}$$

Moreover, $J^\star E$ is also an affine line bundle over $\vec{J}^* E$ and, similarly, $J^{\circledast}E$ is also an affine line bundle over $\vec{J}^{\circledast}E$, whose projections are defined, over each point e of E, by taking the linear part of an affine map. In the twisted case, this bundle projection

$$\eta : J^{\circledast}E \;\longrightarrow\; \vec{J}^{\circledast}E \tag{18}$$

plays an important role because the *hamiltonian* of any classical field theory whose field content is captured by the configuration bundle E is a section

$$\mathcal{H} : \vec{J}^{\circledast}E \;\longrightarrow\; J^{\circledast}E \tag{19}$$

of this projection [4]. In a more physics oriented language, we call $J^{\circledast}E$ the *extended multiphase space* and $\vec{J}^{\circledast}E$ the *ordinary multiphase space* associated with the given configuration bundle E.

Here, the term "multiphase space" is supposed to indicate that the (twisted) cojet bundles $J^{\circledast}E$ and $\vec{J}^{\circledast}E$ carry a multisymplectic structure which is analogous to the symplectic structure on cotangent bundles that qualifies these as candidates for a "phase space" in classical mechanics. Its definition relies on an immediate generalization of the construction of the canonical 1-form on a cotangent bundle known from classical mechanics, namely, the fact that the bundle $\bigwedge^r T^*E$ of r-forms on any manifold E carries a naturally defined "tautological" r-form θ, explicitly given by

$$\begin{array}{l} \theta_\alpha\big(w_1, \ldots w_n\big) \;=\; \alpha\big(T_\alpha\pi(w_1), \ldots, T_\alpha\pi(w_r)\big) \\ \text{for } \;\alpha \in \bigwedge^r T^*E\,,\; w_1, \ldots, w_r \in T_\alpha\big(\bigwedge^r T^*E\big)\;, \end{array} \tag{20}$$

where π denotes the bundle projection from $\bigwedge^r T^*E$ to E, and that when E is the total space of a fiber bundle, then for $0 \leqslant s \leqslant r$, this form restricts to a "tautological" $(r-s)$-horizontal r-form θ on the bundle $\bigwedge_s^r T^*E$ of $(r-s)$-horizontal r-forms on E,[2] which we shall continue to denote by θ, combined with the following canonical isomorphism of vector bundles over E:

$$J^{\circledast}E \;\cong\; \textstyle\bigwedge_1^n T^*E \tag{21}$$

This isomorphism allows us to transfer the form θ, as well as its exterior derivative (up to sign), $\omega = -d\theta$, to forms on $J^{\circledast}E$ which, for the sake of simplicity of notation, we shall continue to denote by θ and by ω, respectively: then θ is called the *multicanonical form* and ω is called the *multisymplectic form* on the extended multiphase space $J^{\circledast}E$. Finally, we introduce the *multicanonical form* $\theta_{\mathcal{H}}$ and the *multisymplectic form* $\omega_{\mathcal{H}}$ on the ordinary multiphase space $\vec{J}^{\circledast}E$ by pulling them back with the hamiltonian $\mathcal{H}$ (see equation (19) above):

$$\theta_{\mathcal{H}} = \mathcal{H}^*\theta \quad , \quad \omega_{\mathcal{H}} = \mathcal{H}^*\omega \tag{22}$$

noting that we still have $\omega = -d\theta$ and $\omega_{\mathcal{H}} = -d\theta_{\mathcal{H}}$.

Within the context outlined above, the traditional method for fixing the dynamics of a specific field theoretical model is by exhibiting its *lagrangian*, which is a homomorphism

$$\mathcal{L} : JE \;\longrightarrow\; \textstyle\bigwedge^n T^*M \tag{23}$$

of fiber bundles over M. The hypothesis that $\mathcal{L}$, when composed with the jet prolongation of a section φ of E, provides an n-form on M, rather than a function, allows us to define the *action functional* S directly by setting, for any compact subset K of M,

$$S_K[\varphi] \;=\; \int_K \mathcal{L}(\varphi, \partial\varphi) \tag{24}$$

without the need of choosing a volume form on space-time: this is supposed to be absorbed in the definition of the lagrangian. Taking its fiber derivative gives rise to the *Legendre transformation*, which comes in two variants: as a homomorphism

$$\mathbb{F}\mathcal{L} : JE \;\longrightarrow\; J^{\circledast}E \tag{25}$$

[2]An r-form on the total space of a fiber bundle is said to be $(r-s)$-horizontal if it vanishes whenever one inserts at least $s+1$ vertical vectors.

or as a homomorphism

$$\vec{\mathbb{F}}\mathcal{L} : JE \longrightarrow \vec{J}^{\circledast} E \tag{26}$$

of fiber bundles over E. Explicitly, given $e \in E$ with $\pi_E(e) = x$ and $u_e \in J_e E \subset L(T_x M, T_e E)$, the latter is defined as the usual fiber derivative of $\mathcal{L}$ at u_e, which is the linear map from $\vec{J}_e E$ to $\bigwedge^n T_x^* M$ given by

$$\vec{\mathbb{F}}\mathcal{L}(u_e) \cdot \vec{u}_e \;=\; \frac{d}{dt}\, \mathcal{L}(u_e + t\vec{u}_e)\Big|_{t=0} \qquad \text{for } \vec{u}_e \in \vec{J}_e E = L(T_x M, V_e E) \;, \tag{27}$$

whereas the former encodes the entire Taylor expansion, up to first order, of $\mathcal{L}$ around u_e along the fibers, which is the affine map from $J_e E$ to $\bigwedge^n T_x^* M$ given by

$$\mathbb{F}\mathcal{L}(u_e) \cdot u'_e \;=\; \mathcal{L}(u_e) \,+\, \frac{d}{dt}\, \mathcal{L}(u_e + t(u'_e - u_e))\Big|_{t=0} \qquad \text{for } u'_e \in J_e E \subset L(T_x M, T_e E) \;. \tag{28}$$

Of course, $\vec{\mathbb{F}}\mathcal{L}$ is just the linear part of $\mathbb{F}\mathcal{L}$, so that it is simply its composition with the bundle projection η from extended to ordinary multiphase space: $\vec{\mathbb{F}}\mathcal{L} = \eta \circ \mathbb{F}\mathcal{L}$. Conversely, we require the hamiltonian $\mathcal{H}$ (see equation (19) above) to satisfy the same sort of composition rule, but in the opposite direction: $\mathbb{F}\mathcal{L} = \mathcal{H} \circ \vec{\mathbb{F}}\mathcal{L}$. In particular, if the lagrangian $\mathcal{L}$ is supposed to be *hyperregular*, which by definition means that $\vec{\mathbb{F}}\mathcal{L}$ should be a global diffeomorphism, then this condition can be used to define the corresponding De Donder – Weyl hamiltonian $\mathcal{H}$ as $\mathcal{H} = \mathbb{F}\mathcal{L} \circ (\vec{\mathbb{F}}\mathcal{L})^{-1}$.

To conclude this discussion, we mention that fixing the configuration bundle E which has been our starting point here, the basic fields of the theory will be sections φ of E: by jet prolongation, they give rise to sections $j\varphi = (\varphi, \partial\varphi)$ of JE and then, by composition with the Legendre transformation $\vec{\mathbb{F}}\mathcal{L}$, to sections $\phi = (\varphi, \pi)$ of $\vec{J}^{\circledast} E$. Moreover, the equations of motion that result from the principle of stationary action, applied to the action functional as defined by equation (24), can be formulated globally (i.e., without resorting to local coordinate expressions) as follows: in the lagrangian framework, the section φ should satisfy the condition

$$(\varphi, \partial\varphi)^* (i_X\, \omega_{\mathcal{L}}) \;=\; 0 \tag{29}$$

for any vector field X on JE which is vertical under the projection to M, or even for any vector field X on JE which is projectable to M, where

$$\omega_{\mathcal{L}} \;=\; (\vec{\mathbb{F}}\mathcal{L})^* \omega_{\mathcal{H}} \;=\; (\mathbb{F}\mathcal{L})^* \omega \tag{30}$$

is the *Poincaré – Cartan form* (Euler – Lagrange equations), whereas in the hamiltonian framework, the section ϕ should satisfy the condition

$$\phi^* (i_X\, \omega_{\mathcal{H}}) \;=\; 0 \tag{31}$$

for any vector field X on $\vec{J}^{\circledast} E$ which is vertical under the projection to M, or even for any vector field X on $\vec{J}^{\circledast} E$ which is projectable to M (De Donder – Weyl equations).

Graphically, we can visualize the situation in terms of the following commutative diagram:

$$
\begin{tikzcd}
 & J^{\circledast}E \arrow[d, shift left, "\eta"] \\
JE \arrow[ur, "\mathbb{F}\mathcal{L}"] \arrow[r, "\vec{\mathbb{F}}\mathcal{L}"'] \arrow[d, "\pi_{JE}"] & \vec{J}^{\circledast}E \arrow[u, shift left, "\mathcal{H}"] \arrow[dl] \\
E \arrow[d, shift left, "\pi_E"] & \\
M \arrow[u, shift left, "\varphi"] \arrow[uu, bend left, "j\varphi"] \arrow[uur, bend right, "\phi"'] &
\end{tikzcd}
\tag{32}
$$

A more detailed treatment of some aspects that, for the sake of brevity, have been omitted here, including explicit expressions in terms of local coordinates, can be found in Ref. [7].

3 Lie groupoids and Lie algebroids

In this section, we shall briefly review the concept of a Lie groupoid, as well as that of an action of a Lie groupoid on a fiber bundle, and then present the construction of the jet groupoid of a Lie groupoid, which is of central importance for applications to field theory. We also discuss the infinitesimal version of these concepts, that is, Lie algebroids, infinitesimal actions and the construction of the jet algebroid of a Lie algebroid. For details and proofs of many of these results, the reader is referred to Ref. [11]: our main goal here is to simplify the notation.

3.1 Lie groupoids

The main feature distinguishing a (Lie) groupoid from a (Lie) group is that, similar to a fiber bundle, it comes with a built-in base space, but it carries two projections to that base space and not just one, called the *source projection* and the *target projection*, so that we can think of the elements of the groupoid as transformations from one given point of the base space to another, and composition of such transformations is allowed only when the target of the first coincides with the source of the second. Formally, a Lie groupoid *over* a manifold M is a manifold[3] G equipped with various structure maps which, when dealing with various Lie groupoids at the same time, we shall decorate with an index G, namely, the *source projection* $\sigma_G : G \longrightarrow M$ and *target projection* $\tau_G : G \longrightarrow M$, which are assumed to be surjective submersions and are sometimes combined into a single map $(\tau_G, \sigma_G) : G \longrightarrow M \times M$ called the *anchor*, the *multiplication map*

$$
\begin{array}{rccc}
\mu_G : & G \times_M G & \longrightarrow & G \\
 & (h,g) & \longmapsto & hg
\end{array}
\tag{33}
$$

defined on the submanifold

$$
G \times_M G \;=\; \big\{ (h,g) \in G \times G \,|\, \sigma_G(h) = \tau_G(g) \big\}
\tag{34}
$$

[3]For some applications, it is necessary to allow the topology of the total space G not to be Hausdorff, but we shall not encounter such types of Lie groupoids here.

of $G \times G$, the *unit map*

$$\begin{array}{rccc} 1_G : & M & \longrightarrow & G \\ & x & \longmapsto & (1_G)_x \end{array} \tag{35}$$

and the *inversion*

$$\begin{array}{rccc} \iota_G : & G & \longrightarrow & G \\ & g & \longmapsto & g^{-1} \end{array} \tag{36}$$

satisfying the usual axioms of group theory, whenever the expressions involved are well-defined, such as the condition of associativity $k(hg) = (kh)g$ for $k, h, g \in G$ such that $\sigma_G(k) = \tau_G(h)$ and $\sigma_G(h) = \tau_G(g)$. For the convenience of the reader, we quickly collect some of the standard terminology in this area:

- given $x \in M$, $G_x = \sigma^{-1}(x)$ is called the *source fiber* over x;
- given $y \in M$, ${}_yG = \tau^{-1}(y)$ is called the *target fiber* over y;
- given $x, y \in M$, ${}_yG_x = G_x \bigcap {}_yG$ is called the *joint fiber* over (y, x);
- given $x \in M$, ${}_xG_x = G_x \bigcap {}_xG$ is a group, called the *isotropy group* or *stability group* or *vertex group* at x;
- given $x \in M$, $G \cdot x = \{y \in M \,|\, {}_yG_x \neq \emptyset\}$ is the *orbit* of x;
- G is said to be *transitive* if there is a unique orbit, namely, all of M, that is, if for any $x \in M$, $G \cdot x = M$;
- G is said to be *totally intransitive* if the orbits are reduced to points, that is if for any $x \in M$, $G \cdot x = \{x\}$, or equivalently, if the source and target projection coincide.

More generally, G is said to be *regular* if the anchor has constant rank. In this case, the subset of G where σ_G and τ_G coincide,

$$G_{\text{iso}} \;=\; \{g \in G \,|\, \sigma_G(g) = \tau_G(g)\} \tag{37}$$

is a totally intransitive Lie subgroupoid of G, called its *isotropy subgroupoid*, and when it is locally trivial (which may not always be the case), it will in fact be a *Lie group bundle* over M. Similarly, in this case, the orbits under the action of G provide a foliation of M called its *characteristic foliation.*

We shall also need the notion of morphism between Lie groupoids: given two Lie groupoids G and G' over the same manifold M,[4] a smooth map $f : G \longrightarrow G'$ is said to be a *morphism*, or *homomorphism*, of Lie groupoids if $\sigma_{G'} \circ f = \sigma_G$, $\tau_{G'} \circ f = \tau_G$ and

$$f(hg) = f(h)f(g) \qquad \text{for } (h, g) \in G \times_M G \;. \tag{38}$$

[4]For the sake of simplicity, we consider here only strict morphisms, that is, morphisms between Lie groupoids over a fixed base manifold M that induce the identity on M. The general case can be reduced to this one by employing the construction of pull-back of Lie groupoids.

Of course, when M reduces to a point, we recover the definition of a Lie group. At the other extreme, we have the following example:

Example 1 Given a manifold M, consider the cartesian product $M \times M$ of two copies of M and define the projection onto the second/first factor as source/target projection ($\sigma(y,x) = x$, $\tau(y,x) = y$), juxtaposition with omission as multiplication ($(z,y)(y,x) = (z,x)$), the diagonal as unit ($1_x = (x,x)$) and switch as inversion $(y,x)^{-1} = (x,y)$. Then $M \times M$ is a Lie groupoid over M, called the *pair groupoid* of M.

Here is another example that will be important in the sequel:

Example 2 Given a manifold M, consider its tangent bundle TM and set

$$GL(TM) \;=\; \dot{\bigcup_{x,y \in M}} {}_yGL(TM)_x \qquad \text{where} \qquad {}_yGL(TM)_x = GL(T_xM, T_yM)$$

is the set of invertible linear transformations from T_xM to T_yM. Then $GL(TM)$, equipped with the obvious operations, is a Lie groupoid over M, called the *linear frame groupoid* of M. Similarly, if in addition we fix a pseudo-riemannian metric g on M and set

$$O(TM,g) \;=\; \dot{\bigcup_{x,y \in M}} {}_yO(TM,g)_x \qquad \text{where} \qquad {}_yO(TM,g)_x = O((T_xM.g_x),(T_yM,g_y))$$

is the set of orthogonal linear transformations from (T_xM, g_x) to (T_yM, g_y), we arrive at what is called the *orthonormal frame groupoid* of M (with respect to g).

In fact, there is a wealth of examples of Lie groupoids, some of which are direct generalizations of Lie groups (such as the frame groupoids of Example 2) and some of which are not (such as the pair groupoid of Example 1). In particular, Example 2 is a special case of a more general construction which associates to each principal bundle over M a transitive Lie groupoid over M called its *gauge groupoid*, here applied to the linear frame bundle of M, as a principal $GL(n,\mathbb{R})$-bundle, where $n = \dim M$.

The insertion of the groupoid concept into group theory becomes apparent when we introduce the notion of bisection of a groupoid, which is the precise analogue of the notion of section of a bundle, the only modification coming from the fact that we now have to deal with two projections rather than one. Namely, just as a (smooth) *section* of a fiber bundle E over a manifold M is a (smooth) map $\varphi : M \longrightarrow E$ such that $\pi_E \circ \varphi = \mathrm{id}_M$, a (smooth) *bisection* of a Lie groupoid G over a manifold M is a (smooth) map $\beta : M \longrightarrow G$ such that $\sigma_G \circ \beta = \mathrm{id}_M$ and $\tau_G \circ \beta \in \mathrm{Diff}(M)$. The point here that the set $\mathrm{Bis}(G)$ of all bisections of a Lie groupoid G is a group, with product defined by

$$(\beta_2\beta_1)(x) \;=\; \beta_2(\tau(\beta_1(x)))\,\beta_1(x) \qquad \text{for } x \in M, \tag{39}$$

so that

$$\tau_G \circ (\beta_2\beta_1) \;=\; (\tau_G \circ \beta_2) \circ (\tau_G \circ \beta_1), \tag{40}$$

with unit given by the unit map of equation (35), which clearly is a bisection, and with inversion defined by

$$\beta^{-1}(x) \;=\; \big(\beta\big((\tau_G \circ \beta)^{-1}(x)\big)\big)^{-1} \qquad \text{for } x \in M. \tag{41}$$

Moreover, at least in the regular case (which is the only one of interest here), $\mathrm{Bis}(G)$ has a natural subgroup, namely the group $\mathrm{Bis}(G_{\mathrm{iso}})$ of bisections (= sections) of the isotropy groupoid of G, and a natural quotient group, namely the group $\mathrm{Diff}^G(M)$ of diffeomorphisms of M obtained by composition of bisections of G with the target projection of G, that is, the image of $\mathrm{Bis}(G)$ under the homomorphism

$$\begin{array}{ccc} \mathrm{Bis}(G) & \longrightarrow & \mathrm{Diff}(M) \\ \beta & \longmapsto & \tau_G \circ \beta \end{array} \tag{42}$$

which fit together into the following exact sequence of groups:

$$\{1\} \longrightarrow \mathrm{Bis}(G_{\mathrm{iso}}) \longrightarrow \mathrm{Bis}(G) \longrightarrow \mathrm{Diff}^G(M) \longrightarrow \{1\}\,. \tag{43}$$

One more concept of central importance that we shall introduce is that of an *action* of a Lie groupoid G on a fiber bundle E, both over the same base manifold M: this is simply a map

$$\begin{array}{cccc} \Phi_E : & G \times_M E & \longrightarrow & E \\ & (g,e) & \longmapsto & g \cdot e \end{array} \tag{44}$$

defined on the submanifold

$$G \times_M E \;=\; \big\{(g,e) \in G \times E \,|\, \sigma_G(g) = \pi_E(e)\big\} \tag{45}$$

of $G \times E$, satisfying $\pi_E \circ \Phi_E = \tau_G \circ \mathrm{pr}_1$ (so that for any $g \in G$, left translation by g becomes a well-defined map from $E_{\sigma_G(g)}$ to $E_{\tau_G(g)}$), together with the usual axioms of an action, adapted to the requirement that the expressions involved should be well-defined, such as the composition rule $h \cdot (g \cdot e) = (hg) \cdot e$ for $h, g \in G$ and $e \in E$ such that $\sigma_G(h) = \tau_G(g)$ and $\sigma_G(g) = \pi_E(e)$. It is then easy to check that any such action induces a representation of the group $\mathrm{Bis}(G)$ of bisections of G by automorphisms of E, that is, a homomorphism

$$\begin{array}{cccc} \Pi_E : & \mathrm{Bis}(G) & \longrightarrow & \mathrm{Aut}(E) \\ & \beta & \longmapsto & \Pi_E(\beta) \end{array} \tag{46}$$

defined by

$$\Pi_E(\beta) \;=\; \Phi_E \circ (\beta \circ \pi_E, \mathrm{id}_E) \tag{47}$$

Note that the automorphism $\Pi_E(\beta)$ of E covers the diffeomorphism $\tau_G \circ \beta$ of M, so in particular $\Pi_E(\beta)$ will be a strict automorphism when β is a bisection (= section) of the isotropy groupoid of G and hence we get the following commutative diagram,

$$\begin{tikzcd} \{1\} \arrow[r] & \mathrm{Aut}_s(E) \arrow[r] & \mathrm{Aut}(E) \arrow[r] & \mathrm{Diff}^E(M) \arrow[r] & \{1\} \\ \{1\} \arrow[r] & \mathrm{Bis}(G_{\mathrm{iso}}) \arrow[r] \arrow[u] & \mathrm{Bis}(G) \arrow[r] \arrow[u] & \mathrm{Diff}^G(M) \arrow[r] \arrow[u] & \{1\} \end{tikzcd} \tag{48}$$

where $\mathrm{Aut}_s(E)$ denotes the group of strict automorphisms of E and $\mathrm{Diff}^E(M)$ denotes the group of diffeomorphisms of M admitting some lift to an automorphism of E.

3.2 The jet groupoid of a Lie groupoid

There are various ways of constructing new Lie groupoids from a given one. As examples, we may mention the *tangent groupoid* and the *action groupoid*, both of which are discussed and widely used in the literature. However, it must be emphasized that both of these constructions imply a change of base manifold. Briefly, given a Lie groupoid G over M, the tangent bundle TG of the total space G is again a Lie groupoid, but over the tangent bundle TM of the base space M rather than over M itself: this is easily seen by defining all structure maps of TG by applying the tangent functor to the corresponding structure maps of G. Similarly, given a Lie groupoid G and a fiber bundle E over M, together with an action of G on E, the submanifold $G \times_M E$ of $G \times E$ introduced above becomes a Lie groupoid, but over the total space E of the given fiber bundle rather than over M (here, the source projection is just projection onto the first factor while the target projection is the action map Φ_E itself). Mathematically, such a change of base space does not present any serious problem, but it does jeopardize the physical interpretation because the whole idea of localizing symmetries that underlies the use of Lie groupoids in field theory refers to space-time and not to any other manifold mathematically constructed from it. Therefore, we should look for constructions that preserve the base space.

This argument leads us directly to the construction of the jet groupoid of a Lie groupoid, which is almost entirely analogous to that of the jet bundle of a fiber bundle, involving just one slight modification:

Proposition 1 *Given a Lie groupoid G over a manifold M, set*

$$JG = \dot{\bigcup_{g \in G}} J_g G$$

where, for $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$,

$$J_g G = \{ u_g \in L(T_x M, T_g G) \,|\, T_g \sigma_G \circ u_g = \mathrm{id}_{T_x M} \text{ and } T_g \tau_G \circ u_g \in GL(T_x M, T_y M) \} \tag{49}$$

which is an open dense subset of the affine space as defined in equation (1), with E replaced by G, π_E replaced by σ_G and e replaced by g.[5] *Then JG is again a Lie groupoid over M, called the* ***jet groupoid*** *of G, with source projection $\sigma_{JG} : G \longrightarrow M$, target projection $\tau_{JG} : G \longrightarrow M$, multiplication map $\mu_{JG} : JG \times_M JG \longrightarrow JG$, unit map $1_{JG} : M \longrightarrow JG$ and inversion $\iota_{JG} : JG \longrightarrow JG$ defined as follows:*

- *for $g \in G$ and $u_g \in J_g G$,*

$$\sigma_{JG}(u_g) = \sigma_G(g) \quad , \quad \tau_{JG}(u_g) = \tau_G(g) \,;$$

- *for $g, h \in G$ with $\tau_G(g) = \sigma_G(h)$, $u_g \in J_g G$ and $v_h \in J_h G$, putting $\sigma_G(g) = x$ and concatenating $v_h \circ (T_g \tau_G \circ u_g)$ and u_g into a single linear map $\big(v_h \circ (T_g \tau_G \circ u_g), u_g\big)$*

[5]The fact that we have slightly modified the meaning of the symbol J when applying it to Lie groupoids, rather than to fiber bundles or, more generally, fibered manifolds, is unlikely to cause any confusion because we are dealing here with distinct categories: equation (1), with the aforementioned substitutions, would only apply if we were to consider G not as a Lie groupoid but just as a fibered manifold with respect to its source projection – which we do not.

from $T_x M$ *into* $T_h G \oplus T_g G$ *and checking that this in fact takes values in the subspace* $T_{(h,g)}(G \times_M G)$,

$$v_h u_g \equiv \mu_{JG}(v_h, u_g) = T_{(h,g)}\mu_G \circ \big(v_h \circ (T_g\tau_G \circ u_g), u_g\big)\,;$$

- *for* $x \in M$,

$$(1_{JG})_x = T_x 1_G\,;$$

- *for* $g \in G$ *and* $u_g \in J_g G$,

$$u_g^{-1} \equiv \iota_{JG}(u_g) = T_g \iota_G \circ u_g \circ (T_g \tau_G \circ u_g)^{-1}.$$

Note that JG *is not only a Lie groupoid over* M *but also a fiber bundle over the total space of the original Lie groupoid* G.

We would like to emphasize that JG does *not* contain G itself as a Lie subgrupoid; rather, G is a quotient of JG under the natural projection

$$\begin{array}{rccc} \pi_{JG}: & JG & \longrightarrow & G \\ & u_g & \longmapsto & g \end{array} \tag{50}$$

which by construction is a morphism of Lie groupoids over M. Similarly, there is a natural projection

$$\begin{array}{rccc} \pi_{JG}^{\mathrm{fr}}: & JG & \longrightarrow & GL(TM) \\ & u_g & \longmapsto & T_g\tau_G \circ u_g \end{array} \tag{51}$$

which again is a morphism of Lie groupoids over M. Taking the fibered product $GL(TM) \times_M G$ of $GL(TM)$ and G over M, which is again a Lie groupoid over M, these two combine into a natural projection

$$\begin{array}{rccc} \pi_{JG}^{\mathrm{fr}} \times_M \pi_{JG}: & JG & \longrightarrow & GL(TM) \times_M G \\ & u_g & \longmapsto & (T_g\tau_G \circ u_g, g) \end{array} \tag{52}$$

which is another morphism of Lie groupoids over M.

One can also define the action of J on morphisms between Lie groupoids (over the same base manifold): given any two Lie groupoids G and G' over M and a morphism $f : G \longrightarrow G'$, one defines a morphism $Jf : JG \longrightarrow JG'$ (sometimes called its jet prolongation or jet extension) by setting

$$J_g f(u_g) \;=\; T_g f \circ u_g \qquad \text{for } g \in G, u_g \in J_g G\,. \tag{53}$$

Obviously, Jf covers f, and one has the following commutative diagram:

$$\begin{tikzcd} JG \arrow[rr, "Jf"] \arrow[d] & & JG' \arrow[d] \\ G \arrow[rr, "f"] \arrow[dr, shift left, "\tau_G"] \arrow[dr, shift right, "\sigma_G"'] & & G' \arrow[dl, shift right, "\sigma_{G'}"'] \arrow[dl, shift left, "\tau_{G'}"] \\ & M & \end{tikzcd} \tag{54}$$

In this way, J becomes a *functor* in the category of Lie groupoids, just as in the category of fiber bundles (always over a fixed base manifold).

Example 3 Up to a canonical isomorphism, the linear frame groupoid $GL(TM)$ is the jet groupoid of the pair groupoid $M \times M$,

$$GL(TM) \cong J(M \times M)\,,$$

and under this isomorphism, the projection in equation (51) above corresponds to the jet prolongation of the anchor map $(\tau_G, \sigma_G) : G \longrightarrow M \times M$.

Just like jet bundles, jet groupoids can be expected to play an important role in differential geometry because they provide the adequate geometric setting for taking derivatives of bisections (which replace sections). Namely, any bisection of G, say β, induces canonically a bisection of JG which will be called its *jet prolongation* or *jet extension* and which we may denote by $j\beta$ or by $(\beta, \partial\beta)$: it is again just a reinterpretation of the tangent map $T\beta$ to β, since $\beta \in \mathrm{Bis}(G)$ implies that, for any $x \in M$, $T_x\beta \in J_{\beta(x)}G$. Obviously, given $g \in G$ with $\sigma_G(g) = x$, every jet $u_g \in J_gG$ can be represented as the derivative at x of some bisection β of G satisfying $\beta(x) = g$, i.e., we can always find β such that $u_g = T_x\beta$, but this does of course not mean that every bisection of JG, as a Lie groupoid over M, can be written as the jet prolongation of some bisection of G: those that can be so written are called *holonomous*, and it is then easy to see that a bisection $\tilde{\beta}$ of JG will be holonomous if and only if $\tilde{\beta} = j\beta$ where $\beta = \pi_{JG} \circ \tilde{\beta}$. This leads us directly to the definition of a special class of Lie subgroupoids of a jet groupoid.[6]

Definition 1 *Let G be a Lie groupoid over a manifold M and $\tilde{G}$ a Lie subgroupoid of its jet groupoid JG that is also a subbundle of JG as a fiber bundle over G. We say that $\tilde{G}$ is* ***holonomous*** *or* ***integrable*** *if it is generated by holonomous bisections, i.e., if for any $g \in G$ with $\sigma_G(g) = x$ and any $u_g \in \tilde{G}_g$, there exists a holonomous bisection $\tilde{\beta} \in \mathrm{Bis}(\tilde{G})$ of $\tilde{G}$ such that $\tilde{\beta}(x) = u_g$.*

The restriction contained in this definition stems from the fact that the group of bisections of $\tilde{G}$ which are also holonomous bisections of JG may be very restricted – to the point of collapsing to the trivial group $\{1\}$.

Here is an important example:

Example 4 Let M be a manifold equipped with a pseudo-riemannian metric $\boldsymbol{g}$ and consider its orthonormal frame groupoid $O(TM, \boldsymbol{g})$ as a Lie subgroupoid of the linear frame groupoid $GL(TM)$ (which according to the previous example is a jet groupoid). Then the group of holonomous bisections of $O(TM, \boldsymbol{g})$ is precisely the isometry group of $(M, \boldsymbol{g})$ and hence $O(TM, \boldsymbol{g})$ is holonomous if and only if $(M, \boldsymbol{g})$ is strongly isotropic; in particular, it must be a space of constant curvature.

In order to handle second order derivatives, we shall need the *second order jet groupoid* J^2G: as in the case of fiber bundles, this can be constructed either directly or else by iteration of the previous construction and subsequent reduction, which occurs in two steps. First, observe that the iterated first order jet groupoid $J(JG)$ admits two natural projections to JG, namely, the standard projection $\pi_{J(JG)} : J(JG) \longrightarrow JG$ and the jet prolongation $J\pi_{JG} : J(JG) \longrightarrow JG$

[6] There is an analogous concept for subbundles of jet bundles which we did not spell out explicitly because we do not need it here.

of the standard projection $\pi_{JG} : JG \longrightarrow G$, explicitly defined as follows: for $g \in G$, $u_g \in J_g G$ and $u'_{u_g} \in J_{u_g}(JG)$,

$$(\pi_{J(JG)})_{u_g}(u'_{u_g}) = u_g \,, \tag{55}$$

whereas

$$(J\pi_{JG})_{u_g}(u'_{u_g}) = T_{u_g}\pi_{JG} \circ u'_{u_g} \,. \tag{56}$$

They fit into the following commutative diagram:

$$\begin{tikzcd} & J(JG) \arrow[dl, "\pi_{J(JG)}"'] \arrow[dr, "J\pi_{JG}"] & \\ JG \arrow[dr, "\pi_{JG}"'] & & JG \arrow[dl, "\pi_{JG}"] \\ & G & \end{tikzcd} \tag{57}$$

The first step of the reduction mentioned above is then again to restrict to the subset of $J(JG)$ where the two projections coincide: for $g \in G$ and $u_g \in J_g G$, define

$$\bar{J}^2_{u_g} G \;=\; \big\{ u'_{u_g} \in J_{u_g}(JG) \,|\, T_{u_g}\pi_{JG} \circ u'_{u_g} = u_g \big\} \,. \tag{58}$$

Taking the disjoint union as u_g varies over JG, this defines what we shall call the *semiholonomic second order jet groupoid* of G, denoted by $\bar{J}^2 G$: since $\pi_{J(JG)}$ and $J\pi_{JG}$ are both morphisms of Lie groupoids over M, it is naturally a Lie groupoid over M and it is also a fiber bundle over JG. The second step consists in decomposing this, as a fiber product of fiber bundles over JG, into a symmetric part and an antisymmetric part: the former is precisely $J^2 G$ whereas the latter does not seem to play any significant role in the theory; we shall therefore not pursue this decomposition any further. But what will be important in the sequel is that, once again, given any bisection $\tilde{\beta}$ of JG, its jet prolongation $j\tilde{\beta}$ will be a bisection of $J(JG)$ which will take values in $\bar{J}^2 G$ if and only if $\tilde{\beta}$ is holonomous, in which case $\tilde{\beta} = j\beta$ and $j\tilde{\beta} = j^2\beta$ where $\beta = \pi_{JG} \circ \tilde{\beta}$. Using the terminology introduced in Definition 1, we may thus characterize $J^2 G$ as the unique maximal holonomous Lie subgroupoid of $\bar{J}^2 G$, i.e., that Lie subgroupoid of $\bar{J}^2 G$ whose bisections are precisely the holonomous bisections of $\bar{J}^2 G$.

3.3 Lie algebroids

Just as Lie algebras are the infinitesimal version of Lie groups, Lie algebroids are the infinitesimal version of Lie groupoids. Formally, a Lie algebroid *over* a manifold M is a vector bundle $\mathfrak{g}$ over M endowed with two structure maps which, when dealing with various Lie algebroids at the same time, we shall decorate with an index $\mathfrak{g}$, namely, the *anchor* $\alpha_{\mathfrak{g}} : \mathfrak{g} \longrightarrow TM$, which is required to be a homomorphism of vector bundles over M and, by push-forward of sections, induces a homomorphism $\alpha_{\mathfrak{g}} : \Gamma(\mathfrak{g}) \longrightarrow \mathfrak{X}(M)$ of modules over the function ring $C^\infty(M)$ (we follow the common abuse of notation of denoting both by the same symbol), and the *bracket*

$$[.\,,.]_{\mathfrak{g}} : \;\Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \;\;\longrightarrow\;\; \Gamma(\mathfrak{g}) \tag{59}$$

satisfying the usual axioms of Lie algebra theory (bilinearity over $\mathbb{R}$, antisymmetry and the Jacobi identity), together with the Leibniz identity

$$[X, fY]_{\mathfrak{g}} \;=\; f[X,Y]_{\mathfrak{g}} \,+\, (L_{\alpha_{\mathfrak{g}}(X)} f)\, Y \qquad \text{for } f \in C^\infty(M),\, X, Y \in \Gamma(\mathfrak{g})\,. \tag{60}$$

Once again, we gather some standard terminology. For example, $\mathfrak{g}$ is said to be *transitive* if the anchor is surjective and *totally intransitive* if it is zero. More generally, $\mathfrak{g}$ is said to be *regular* if the anchor has constant rank. In this case, the kernel of the anchor,

$$\mathfrak{g}_{\mathrm{iso}} \;=\; \ker \alpha_{\mathfrak{g}} \tag{61}$$

is a totally intransitive Lie subalgebroid of $\mathfrak{g}$, called its *isotropy algebroid*, and when it is locally trivial (which may not always be the case), it will in fact be a *Lie algebra bundle* over M. Similarly, in this case, the image of the anchor,

$$\mathfrak{X}^{\mathfrak{g}}(M) \;=\; \operatorname{im} \alpha_{\mathfrak{g}} \tag{62}$$

is an involutive distribution on M called its *characteristic distribution*.

We shall skip the definition of morphism between Lie algebroids (over the same manifold), which is the obvious one, as well as the discussion of examples, and just mention that the insertion of the Lie algebroid concept into Lie algebra theory is even more direct than in the group case, since the space $\Gamma(\mathfrak{g})$ of sections of a Lie algebroid $\mathfrak{g}$ is a Lie algebra by definition. Moreover, at least in the regular case (which is the only one of interest here), $\Gamma(\mathfrak{g})$ has a natural Lie subalgebra, namely the Lie algebra $\Gamma(\mathfrak{g}_{\mathrm{iso}})$ of sections of the isotropy algebroid of $\mathfrak{g}$, and a natural quotient Lie algebra, namely the Lie algebra $\mathfrak{X}^{\mathfrak{g}}(M)$ of sections of the characteristic distribution on M, which fit together into the following exact sequence of Lie algebras:

$$\{0\} \;\longrightarrow\; \Gamma(\mathfrak{g}_{\mathrm{iso}}) \;\longrightarrow\; \Gamma(\mathfrak{g}) \;\longrightarrow\; \mathfrak{X}^{\mathfrak{g}}(M) \;\longrightarrow\; \{0\}\,. \tag{63}$$

Next, let us pass to the construction of the Lie algebroid of a Lie groupoid, which once again follows closely that of the Lie algebra of a Lie group; the same goes for the exponential. Given a Lie groupoid G over a manifold M, we use the unit map 1_G to consider M as an embedded submanifold of G and restrict the tangent maps to the source and target projections to this submanifold, which provides us with two homomorphisms of vector bundles over M that we can put into the following diagram:

$$\begin{tikzcd} TG|_M \arrow[rr, shift left, "T\sigma|_M"] \arrow[rr, shift right, "T\tau|_M"'] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd} \tag{64}$$

As a vector bundle over M, the corresponding Lie algebroid $\mathfrak{g}$ is then defined as the restriction to M of the vertical bundle with respect to the source projection:

$$\mathfrak{g} \;=\; (V^{\sigma}G)|_M \;=\; \ker T\sigma|_M\,. \tag{65}$$

More explicitly, this means that, for any $x \in M$, $\mathfrak{g}_x$ is the tangent space to the source fiber at x:

$$\mathfrak{g}_x \;=\; V_x^{\sigma}G \;=\; T_x(G_x)\,. \tag{66}$$

The anchor of $\mathfrak{g}$ is then defined as the restriction of the target projection:

$$\alpha_{\mathfrak{g}} \;=\; \left(T\tau_G|_M\right)\big|_{\mathfrak{g}}\,. \tag{67}$$

To construct the bracket between sections of $\mathfrak{g}$, we note that given g in G with $\sigma_G(g) = x$ and $\tau_G(g) = y$, right translation by g is a diffeomorphism from the source fiber at y to the source fiber at x:

$$\begin{array}{rccc} R_g: & G_y & \longrightarrow & G_x \\ & h & \longmapsto & hg \end{array} \tag{68}$$

Thus its derivative at the point $h \in G_y$ is a linear isomorphism $T_h R_g : T_h(G_y) \longrightarrow T_{hg}(G_x)$, and hence given a vector field Z on G that is σ_G-vertical, $Z \in \Gamma(V^\sigma G)$, it makes sense to say that Z is *right invariant* if, for any $(h, g) \in G \times_M G$,

$$T_h R_g(Z(h)) \;=\; Z(hg)\,.$$

Now observe that (a) the space $\mathfrak{X}^{ri}(G)$ of right invariant vector fields on G is a Lie subalgebra of the space $\mathfrak{X}(G)$ of all vector fields on G and (b) restriction to M provides a linear isomorphism from $\mathfrak{X}^{ri}(G)$ to the space $\Gamma(\mathfrak{g})$ of sections of $\mathfrak{g}$ whose inverse can be described explicitly as follows: given $X \in \Gamma(\mathfrak{g})$, the corresponding right invariant vector field $X^r \in \mathfrak{X}^{ri}(G)$ is defined by

$$X^r(g) \;=\; T_{\tau_G(g)} R_g\big(X(\tau_G(g))\big)\,.$$

This linear isomorphism is used to transfer the structure of Lie algebra from $\mathfrak{X}^{ri}(G)$ to $\Gamma(\mathfrak{g})$. The same idea is used to construct the *exponential*, as a map

$$\exp:\; \Gamma(\mathfrak{g}) \;\longrightarrow\; \mathrm{Bis}(G) \tag{69}$$

defined by taking the flow F_{X^r} of the right invariant vector field X^r on G corresponding to X at time 1:

$$\exp(X)(x) \;=\; F_{X^r}(1, 1_x) \qquad \text{for } x \in M\,.$$

Then, more generally,

$$F_{X^r}(t, g) \;=\; \exp(tX)(y)\, g \qquad \text{for } t \in \mathbb{R},\, g \in {}_yG_x\,, \tag{70}$$

and conversely,

$$\exp(tX)(x)\big|_{t=0} \;=\; 1_x \;\;,\;\; \frac{d}{dt}\exp(tX)(x)\Big|_{t=0} \;=\; X(x) \qquad \text{for } x \in M. \tag{71}$$

Continuing the analogies between Lie group theory and Lie groupoid theory, there is also the concept of an *infinitesimal action* of a Lie algebroid $\mathfrak{g}$ on a fiber bundle E, both over the same base manifold M: this is defined to be a linear map

$$\begin{array}{rccc} \dot{\Phi}_E: & \Gamma(\mathfrak{g}) & \longrightarrow & \mathfrak{X}^P(E) \\ & X & \longmapsto & X_E \end{array} \tag{72}$$

where $\mathfrak{X}^P(E)$ denotes the Lie algebra of projectable vector fields on E,[7] which is compatible with the structures involved: it is linear over the pertinent function rings, i.e.,

$$(fX)_E \;=\; (f \circ \pi)\, X_E \qquad \text{for } f \in C^\infty(M),\, X \in \Gamma(\mathfrak{g})\,, \tag{73}$$

[7] A vector field X_E on the total space E of a fiber bundle is said to be projectable if for any two points e_1, e_2 in the same fiber, we have $T_{e_1}\pi_E(X_E(e_1)) = T_{e_2}\pi_E(X_E(e_2))$, that is, if there exists a (necessarily unique) vector field X_M on the base space M to which it is π_E-related: one then says that X_E projects to X_M. Vertical vector fields are those that project to 0.

takes the anchor to the projection, i.e.,

$$X_M \;=\; \alpha(X) \qquad \text{for } X \in \Gamma(\mathfrak{g})\,, \tag{74}$$

and preserves brackets, i.e., is a homomorphism of Lie algebras; we call X_E the *fundamental vector field* associated to X. The terminology is justified by noting that if $\mathfrak{g}$ is the Lie algebroid of a Lie groupoid G, then an action of G on E induces an infinitesimal action of $\mathfrak{g}$ on E defined as follows: given $X \in \Gamma(\mathfrak{g})$ and recalling that, for any $x \in M$,

$$X(x) \;=\; \frac{d}{dt}\,\exp(tX)(x)\,\Big|_{t=0}, \tag{75}$$

we have, for any $e \in E$ with $x = \pi_E(e) \in M$,

$$X_E(e) \;=\; \frac{d}{dt}\,\exp(tX)(x)\cdot e\,\Big|_{t=0}. \tag{76}$$

In fact, the above definition of infinitesimal action provides nothing more and nothing less than a representation of the Lie algebra of sections of the Lie algebroid $\mathfrak{g}$ by projectable vector fields on E which is the formal derivative of the representation Π_E of the group of bisections of the Lie groupoid G by automorphisms of E in equations (46) and (47). Note that the fundamental vector field X_E associated to a section X of the isotropy algebroid of $\mathfrak{g}$ will be vertical and hence we get the following commutative diagram,

$$\begin{tikzcd}
\{0\} \arrow[r] & \mathfrak{X}_V(E) \arrow[r] & \mathfrak{X}_P(E) \arrow[r] & \mathfrak{X}^E(M) \arrow[r] & \{0\} \\
\{0\} \arrow[r] & \Gamma(\mathfrak{g}_{\mathrm{iso}}) \arrow[r] \arrow[u] & \Gamma(\mathfrak{g}) \arrow[r] \arrow[u] & \mathfrak{X}^{\mathfrak{g}}(M) \arrow[r] \arrow[u] & \{0\}
\end{tikzcd} \tag{77}$$

where $\mathfrak{X}^V(E)$ denotes the Lie algebra of vertical vector fields on E and $\mathfrak{X}^E(M)$ denotes the algebra of vector fields on M admitting some lift to a projectable vector field on E.

3.4 The jet algebroid of a Lie algebroid

As in the case of Lie groupoids, there are various ways of constructing new Lie algebroids from a given one. Here, we want to mention just one of them, namely that of the jet algebroid of a Lie algebroid. To do so, we use the functor J on fiber bundles and observe that, for any manifold M, the jet bundle of a fiber bundle over M carrying some additional structure will in many cases inherit that additional structure. In particular, if E is a vector bundle over M, its jet bundle JE is again a vector bundle over M: this can be most easily seen by writing points in JE as values of jets of sections of E and defining a linear combination of points in JE over the same point of M as the value of the jet of the corresponding linear combination of sections of E: this is the necessary and sufficient condition for the jet prolongation of sections (from sections of E to sections of JE) to be a linear map $j : \Gamma(E) \longrightarrow \Gamma(JE)$.

Proposition 2 *Given a Lie algebroid $\mathfrak{g}$ over a manifold M with anchor $\alpha_{\mathfrak{g}}$ and bracket $[.\,,.]_{\mathfrak{g}}$, its jet bundle $J\mathfrak{g}$ is again a Lie algebroid over M, with anchor $\alpha_{J\mathfrak{g}}$ and bracket $[.\,,.]_{J\mathfrak{g}}$ defined so that the jet prolongation of sections (from sections of $\mathfrak{g}$ to sections of $J\mathfrak{g}$) preserves the anchor and the bracket:*

$$\alpha_{J\mathfrak{g}}(j\xi) \;=\; \alpha_{\mathfrak{g}}(\xi) \quad , \quad [j\xi, j\eta]_{J\mathfrak{g}} \;=\; j([\xi,\eta]_{\mathfrak{g}}) \qquad \textit{for } \xi,\eta \in \Gamma(\mathfrak{g})\,.$$

We note that this condition of compatibility with the procedure of taking the jet prolongation of sections fixes the anchor and the bracket in $J\mathfrak{g}$ uniquely. To prove existence, one can employ various methods, but the most transparent one of them, at least for physicists, is in terms of the corresponding structure functions, using the fact that the anchors and brackets, in $\mathfrak{g}$ as well as in $J\mathfrak{g}$, are differential operators in each argument, and hence local. (More precisely, the anchors are differential operators of order 0, since they result from push-forward of sections by a vector bundle homomorphism, while the brackets are bidifferential operators of order 1, since they satisfy a Leibniz rule, as in equation (60), in each factor.) Namely, given local coordinates x^μ of M and a basis of local sections T_a of $\mathfrak{g}$, together with the induced basis of local sections (T_a, T_a^μ) of $J\mathfrak{g}$, we have that if ξ is a section of $\mathfrak{g}$, locally represented as

$$\xi = \xi^a T_a,$$

then $j\xi$ is a section of $J\mathfrak{g}$ locally represented as

$$j\xi = \xi^a T_a + \partial_\mu \xi^a \, T_a^\mu.$$

Assuming that the anchor and the bracket of $\mathfrak{g}$ can be written in terms of structure functions f_a^μ and f_{ab}^c, according to

$$\alpha_{\mathfrak{g}}(T_a) \;=\; f_a^\mu \, \partial_\mu \tag{78}$$

and

$$[T_a, T_b]_{\mathfrak{g}} \;=\; f_{ab}^c T_c \tag{79}$$

the anchor and the bracket of $J\mathfrak{g}$ are given by

$$\begin{aligned} \alpha_{J\mathfrak{g}}(T_a) \;&=\; f_a^\mu \, \partial_\mu \\ \alpha_{J\mathfrak{g}}(T_a^\mu) \;&=\; 0 \end{aligned} \tag{80}$$

and

$$\begin{aligned} [T_a, T_b]_{J\mathfrak{g}} \;&=\; f_{ab}^c T_c \,+\, \partial_\mu f_{ab}^c \, T_c^\mu \\ [T_a, T_b^\mu]_{J\mathfrak{g}} \;&=\; f_{ab}^c T_c^\mu \,+\, \partial_\nu f_a^\mu \, T_b^\nu \\ [T_a^\mu, T_b^\nu]_{J\mathfrak{g}} \;&=\; f_a^\nu T_b^\mu \,-\, f_b^\nu T_a^\mu \end{aligned} \tag{81}$$

We leave it to the reader to convince himself that, up to a canonical isomorphism, the functor J commutes with the process of passing from a Lie groupoid to its corresponding Lie algebroid, that is, if $\mathfrak{g}$ is the Lie algebroid associated to the Lie groupoid G, then $J\mathfrak{g}$ is the Lie algebroid associated to the Lie groupoid JG.

4 Induced actions

Starting from a given action of a Lie groupoid on a fiber bundle, as we shall in this section show how to construct "induced actions" of certain other Lie groupoids, derived from the original one, on certain other fiber bundles, derived from the original one: this is an essential technical feature needed to make the theory work. Throughout the section, we maintain the notation used before: G will be a Lie groupoid over a manifold M, with source projection σ_G and target projection τ_G, multiplication μ_G, unit map 1_G and inversion ι_G, and E will be a fiber bundle

over the same manifold M, with projection π_E, endowed with an action Φ_E as in equation (44). Note that under these circumstances, a groupoid element $g \in G$ with source $\sigma_G(g) = x$ and target $\tau_G(g) = y$ will provide a diffeomorphism $L_g : E_x \longrightarrow E_y$ called *left translation* by g.

Our first and most elementary example of such an induced action is that of the original Lie groupoid G on the vertical bundle VE of E,

$$\begin{array}{rccc} \Phi_{VE}: & G \times_M VE & \longrightarrow & VE \\ & (g, v_e) & \longmapsto & g \cdot v_e \end{array} \tag{82}$$

defined by requiring left translation by g in VE to be simply the derivative of left translation by g in E, i.e., given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi_E(e) = x$ and a vertical vector $v_e \in V_e E$, we have

$$g \cdot v_e \;=\; T_e L_g(v_e)\,. \tag{83}$$

This means that given a vertical curve $t \mapsto e(t)$ in E passing through the point e (i.e., $e(0) = e$), we have

$$g \cdot \frac{d}{dt}\, e(t)\Big|_{t=0} \;=\; \frac{d}{dt}\, g \cdot e(t)\Big|_{t=0}. \tag{84}$$

An important property of the action (82) is that it respects the structure of VE as a vector bundle over E, since (a) it covers the original action, i.e., the diagram

$$\begin{tikzcd} G \times_M VE \arrow[r, "\Phi_{VE}"] \arrow[d] & VE \arrow[d] \\ G \times_M E \arrow[r, "\Phi_E"'] & E \end{tikzcd} \tag{85}$$

commutes, and (b) for any $g \in G$, left translation by g is well defined on the whole vertical space $V_e E$ provided that $\pi_E(e) = \sigma_G(g)$ (and otherwise is not well defined at any point in $V_e E$), its restriction to this space being a linear transformation $L_g : V_e E \longrightarrow V_{g \cdot e} E$, since it is the tangent map at e to left translation $L_g : E_x \longrightarrow E_y$.

Our next example of an induced action is functorial. Namely, applying the jet functor to all structural maps of the original action, we obtain an action of the jet groupoid JG of G on the jet bundle JE of E,

$$\begin{array}{rccc} \Phi_{JE}: & JG \times_M JE & \longrightarrow & JE \\ & (u_g, u_e) & \longmapsto & u_g \cdot u_e \end{array} \tag{86}$$

defined as follows: given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi_E(e) = x$ and jets $u_g \in J_g G \subset L(T_x M, T_g G)$ and $u_e \in J_e E \subset L(T_x M, T_e E)$, we concatenate both in a linear map $(u_g, u_e) \in L(T_x M, T_g G \oplus T_e E)$ (which actually takes values in $L(T_x M, T_{(g,e)}(G \times_M E))$ since $T_x \sigma_G \circ u_g = \mathrm{id}_{T_x M} = T_e \pi_E \circ u_e$) and compose:

$$u_g \cdot u_e \;=\; T_{(g,e)} \Phi_E \circ (u_g, u_e) \circ (T_g \tau \circ u_g)^{-1}. \tag{87}$$

Essentially the same procedure also provides an action of the jet groupoid JG of G on the linearized jet bundle $\vec{J}E$ of E,

$$\begin{array}{rccc} \Phi_{\vec{J}E}: & JG \times_M \vec{J}E & \longrightarrow & \vec{J}E \\ & (u_g, \vec{u}_e) & \longmapsto & u_g \cdot \vec{u}_e \end{array} \tag{88}$$

Unlike the previous one, this admits a simplification because it factorizes through the morphism (52) of Lie groupoids to yield an action of the Lie groupoid $GL(TM) \times_M G$ on the linearized jet bundle $\vec{J}E$ of E,

$$\begin{array}{ccc} \big(GL(TM) \times_M G\big) \times_M \vec{J}E & \longrightarrow & \vec{J}E \\ \big((a,g), \vec{u}_e\big) & \longmapsto & (a,g) \cdot \vec{u}_e \end{array} \tag{89}$$

This action is suggested by the isomorphism

$$\vec{J}E \;\cong\; \pi_E^*\big(T^*M\big) \otimes VE \tag{90}$$

(see equation (3)), together with the fact that the tangent bundle and the cotangent bundle of M are endowed with natural actions of the linear frame groupoid $GL(TM)$ and the vertical bundle of E is endowed with the induced action of G as explained above. However, it should be noted that all the groupoids involved are groupoids over M while the isomorphism (90) is one of vector bundles over E. Therefore, it is worthwhile specifying that the action (89) is explicitly defined as follows: given $(a,g) \in GL(TM) \times_M G$ with $\sigma_{GL(TM)}(a) = x = \sigma_G(g)$ and $\tau_{GL(TM)}(a) = y = \tau_G(g)$, $e \in E$ with $\pi_E(e) = x$ and $\vec{u}_e \in \vec{J}_e E = L(T_x M, V_e E)$, we obtain $(a,g) \cdot \vec{u}_e \in \vec{J}_{g\cdot e} E = L(T_y M, V_{g\cdot e} E)$ by composition:

$$(a,g) \cdot \vec{u}_e \;=\; L_g \circ \vec{u}_e \circ a^{-1} \tag{91}$$

An important property of the actions (86) and (88) is that they respect the structure of JE as an affine bundle and of $\vec{J}E$ as a vector bundle over E, since (a) they cover the original action, i.e., the diagrams

$$\begin{tikzcd} JG \times_M JE \arrow[r, "\Phi_{JE}"] \arrow[d] & JE \arrow[d] \\ G \times_M E \arrow[r, "\Phi_E"'] & E \end{tikzcd} \tag{92}$$

and

$$\begin{tikzcd} JG \times_M \vec{J}E \arrow[r, "\Phi_{\vec{J}E}"] \arrow[d] & \vec{J}E \arrow[d] \\ G \times_M E \arrow[r, "\Phi_E"'] & E \end{tikzcd} \tag{93}$$

commute, and (b) for any $g \in G$ and $u_g \in J_g G$, left translation by u_g is well defined on the whole jet space $J_e E$ and on the whole linearized jet space $\vec{J}_e E$ provided that $\pi_E(e) = \sigma_G(g)$ (and otherwise is not well defined at point in $J_e E$ or $\vec{J}_e E$), its restriction to these spaces being an affine transformation $L_{u_g} : J_e E \longrightarrow J_{g\cdot e} E$ or a linear transformation $L_{u_g} : \vec{J}_e E \longrightarrow \vec{J}_{g\cdot e} E$.

The compatibility of the actions (86) and (88) with the structure of JE as an affine bundle and of $\vec{J}E$ as a vector bundle over E allows us to transfer these actions of jets to cojets (ordinary or twisted), by dualization. Concentrating on the twisted case, which is the more important one for the applications we have in mind, we obtain an action

$$\begin{array}{rccc} \Phi_{J^{\circledast}E} : & JG \times_M J^{\circledast}E & \longrightarrow & J^{\circledast}E \\ & (u_g, z_e) & \longmapsto & u_g \cdot z_e \end{array} \tag{94}$$

of JG on $J^{\circledast}E$ defined as follows: given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi_E(e) = x$, $u_g \in J_g G$, $z_e \in J^{\circledast}_e E$ and $u_{g\cdot e} \in J_{g\cdot e}E$,

$$\langle u_g \cdot z_e \, , \, u_{g\cdot e} \rangle \;=\; (T_g \tau_G \circ u_g)^{-1\,*} \langle z_e \, , \, u_g^{-1} \cdot u_{g\cdot e} \rangle \,. \tag{95}$$

In other words, we require the following diagram to commute:

$$\begin{tikzcd} J_e E \arrow[r, "L_{u_g}"] \arrow[d, "z_e"'] & J_{g\cdot e}E \arrow[d, "u_g\cdot z_e"] \\ \bigwedge^n T_x^* M \arrow[r, "(T_g\tau_G \circ u_g)^{-1\,*}"'] & \bigwedge^n T_y^* M \end{tikzcd} \tag{96}$$

Similarly, we obtain an action

$$\begin{array}{rccc} \Phi_{\vec{J}^{\circledast}E} : & JG \times_M \vec{J}^{\circledast}E & \longrightarrow & \vec{J}^{\circledast}E \\ & (u_g, \vec{z}_e) & \longmapsto & u_g \cdot \vec{z}_e \end{array} \tag{97}$$

of JG on $\vec{J}^{\circledast}E$ that factorizes through the composition of the morphism (52) of Lie groupoids to yield an action

$$\begin{array}{ccc} \big(GL(TM) \times_M G\big) \times_M \vec{J}^{\circledast}E & \longrightarrow & \vec{J}^{\circledast}E \\ \big((a,g), \vec{z}_e\big) & \longmapsto & (a,g) \cdot \vec{z}_e \end{array} \tag{98}$$

of $GL(TM) \times_M G$ on $\vec{J}^{\circledast}E$ defined as follows: given $(a,g) \in GL(TM) \times_M G$ with $\sigma_{GL(TM)}(a) = x = \sigma_G(g)$ and $\tau_{GL(TM)}(a) = y = \tau_G(g)$, $e \in E$ with $\pi_E(e) = x$, $\vec{z}_e \in \vec{J}^{\circledast}_e E$ and $\vec{u}_{g\cdot e} \in \vec{J}_{g\cdot e}E$,

$$\langle (a,g) \cdot \vec{z}_e \, , \, \vec{u}_{g\cdot e} \rangle \;=\; a^{-1\,*} \langle \vec{z}_e \, , \, (a,g)^{-1} \cdot \vec{u}_{g\cdot e} \rangle, \tag{99}$$

In other words, we require the following diagram to commute:

$$\begin{tikzcd} \vec{J}_e E \arrow[r, "L_{(a,g)}"] \arrow[d, "\vec{z}_e"'] & \vec{J}_{g\cdot e}E \arrow[d, "(a,g)\cdot\vec{z}_e"] \\ \bigwedge^n T_x^* M \arrow[r, "a^{-1\,*}"'] & \bigwedge^n T_y^* M \end{tikzcd} \tag{100}$$

All these actions again satisfy the property of compatibility with the structure of the bundles involved as vector bundles over E.

Passing to our next example, which is once again functorial, let us now apply the tangent functor T to all structural maps of the original action to obtain an action of the tangent groupoid TG of G on the tangent bundle TE of E,

$$\begin{array}{ccc} TG \times_{TM} TE & \longrightarrow & TE \\ (v_g, v_e) & \longmapsto & v_g \cdot v_e \end{array} \tag{101}$$

where we have used the canonical identification of $T\big(G \times_M E\big)$ with $TG \times_{TM} TE$, under which a pair of vectors $(v_g, v_e) \in T_g G \oplus T_e E$ belongs to the subspace $T_{(g,e)}\big(G \times_M E\big) = (TG \times_{TM} TE)_{(g,e)}$

if and only if its two components are related according to $T_g\sigma_G(v_g) = T_e\pi_E(v_e)$, and in this case,

$$v_g \cdot v_e \;=\; T_{(g,e)}\Phi_E(v_g, v_e)\,. \tag{102}$$

Now even though this action still covers the original one, i.e., the diagram

$$\begin{tikzcd} TG \times_{TM} TE \arrow[r, "T\Phi_E"] \arrow[d] & TE \arrow[d] \\ G \times_M E \arrow[r, "\Phi_E"'] & E \end{tikzcd} \tag{103}$$

commutes, the problem is that it involves a change of base space, from M to TM, and as a result it does not respect the structure of TE as a vector bundle over E. Namely, given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi_E(e) = x$ and $v_g \in T_gG$ with $T_g\sigma_G(v_g) = v_x \in T_xM$ and $T_g\tau_G(v_g) = v_y \in T_yM$, left translation by v_g is not well defined on the whole tangent space T_eE, but only on its affine subspace $(T_e\pi_E)^{-1}(v_x)$, and its restriction to this subspace is an affine transformation $L_{v_g} : (T_e\pi_E)^{-1}(v_x) \longrightarrow (T_{g\cdot e}\pi_E)^{-1}(v_y)$.

This is a serious defect because it prevents the transfer of this action to cotangent vectors or, more generally, tensors on E and thus makes it almost useless.

Fortunately, and that is perhaps the central message of this paper, there is a way out of this impasse: it consists in replacing the tangent groupoid TG by the jet groupoid JG. In fact, as we will show now, there is a natural induced action of the jet groupoid JG of G on the tangent bundle TE of E,

$$\begin{array}{rccc} \Phi_{TE}: & JG \times_M TE & \longrightarrow & TE \\ & (u_g, v_e) & \longmapsto & u_g \cdot v_e \end{array} \tag{104}$$

defined as follows: given $g \in G$ and $e \in E$ with $\sigma_G(g) = \pi_E(e)$, $u_g \in J_gG$ and $v_e \in T_eE$,

$$u_g \cdot v_e \;=\; T_{(g,e)}\Phi_E\big((u_g \circ T_e\pi_E)(v_e), v_e\big). \tag{105}$$

This prescription is less obvious than the previous ones because it mixes the two functors J and T, so it may be worthwhile to check explicitly that it does indeed define an action. To this end, we note first that, with g, e, u_g and v_e as before,

$$\begin{aligned} T_{g\cdot e}\pi_E(u_g \cdot v_e) \;&=\; T_{(g,e)}(\pi_E \circ \Phi_E)\big((u_g \circ T_e\pi_E)(v_e), v_e\big) \\ &=\; T_{(g,e)}(\tau_G \circ \mathrm{pr}_1)\big((u_g \circ T_e\pi_E)(v_e), v_e\big) \\ &=\; T_g\tau_G\big((u_g \circ T_e\pi_E)(v_e)\big) \;=\; (T_g\tau_G \circ u_g)(T_e\pi_E(v_e))\,. \end{aligned} \tag{106}$$

Therefore, given $g, h \in G$ and $e \in E$ with $\sigma_G(g) = \pi_E(e)$, $\sigma_G(h) = \tau_G(g) = \pi_E(g \cdot e)$ and $u_g \in J_gG$, $u_h \in J_hG$, $v_e \in T_eE$,

$$\begin{aligned} u_h \cdot (u_g \cdot v_e) \;&=\; T_{(h,g\cdot e)}\Phi_E\big((u_h \circ T_{g\cdot e}\pi_E)(u_g \cdot v_e)\,, u_g \cdot v_e\big) \\ &=\; T_{(h,g\cdot e)}\Phi_E\big((u_h \circ (T_g\tau_G \circ u_g) \circ T_e\pi_E)(v_e)\,, T_{(g,e)}\Phi_E\big((u_g \circ T_e\pi_E)(v_e), v_e\big) \\ &=\; T_{(h,g,e)}\big(\Phi_E \circ (\mathrm{id}_G \times \Phi_E)\big)\big((u_h \circ (T_g\tau_G \circ u_g) \circ T_e\pi_E)(v_e)\,, (u_g \circ T_e\pi_E)(v_e)\,, v_e\big) \\ &=\; T_{(h,g,e)}\big(\Phi_E \circ (\mu_G \times \mathrm{id}_E)\big)\big((u_h \circ (T_g\tau_G \circ u_g) \circ T_e\pi_E)(v_e)\,, (u_g \circ T_e\pi_E)(v_e)\,, v_e\big) \\ &=\; T_{(hg,e)}\Phi_E\big(\big(T_{(h,g)}\mu_G \circ \big(u_h \circ (T_g\tau_G \circ u_g), u_g\big) \circ T_e\pi_E\big)(v_e)\,, v_e\big) \\ &=\; T_{(hg,e)}\Phi_E\big((u_h u_g \circ T_e\pi_E)(v_e), v_e\big) \\ &=\; (u_h u_g) \cdot v_e\,. \end{aligned}$$

Similarly, given $g \in G$ and $e \in E$ with $\sigma_G(g) = x = \pi_E(e)$, $u_g \in J_gG$ and $v_e \in T_eE$,

$$\begin{aligned} 1_{JG,x} \cdot v_e &= T_{1_{G,x}}\Phi_E\big((T_x 1_G \circ T_e\pi_E)(v_e)\,,v_e\big) \\ &= T_e\big(\Phi_E \circ (1_G \circ \pi_E\,,\mathrm{id}_E)\big)(v_e) \\ &= T_e\,\mathrm{id}_E(v_e) \;=\; v_e\,. \end{aligned}$$

Moroever, it follows from equation (106) that this action preserves the vertical bundle VE, and comparing equation (83) or (84) with equation (105) shows that its restriction to VE factorizes through the projection from JG to G so as to yield the action (82) introduced above.

A fundamental property of the action (104) is that this one does respect the structure of TE as a vector bundle over E, since (a) it covers the original action, i.e., the diagram

$$\begin{tikzcd} JG \times_M TE \arrow[r, "\Phi_{TE}"] \arrow[d] & TE \arrow[d] \\ G \times_M E \arrow[r, "\Phi_E"'] & E \end{tikzcd} \tag{107}$$

commutes, and (b) for any $g \in G$ and $u_g \in J_gG$, left translation by u_g is well defined on the whole tangent space T_eE provided that $\pi_E(e) = \sigma_G(g)$ (and otherwise is not well defined at any point in T_eE), its restriction to this space being a linear transformation $L_{u_g} : T_eE \longrightarrow T_{g\cdot e}E$, because it is the composition of two linear maps:

$$L_{u_g} \;=\; T_{(g,e)}\Phi_E \circ \big(u_g \circ T_e\pi_E\,,\mathrm{id}_{T_eE}\big).$$

And finally, we note that equation (106) states that the action (104) also covers the natural action of $GL(TM)$ on TM, i.e., the diagram

$$\begin{tikzcd} JG \times_M TE \arrow[r, "\Phi_{TE}"] \arrow[d] & TE \arrow[d] \\ GL(TM) \times_M TM \arrow[r] & TM \end{tikzcd} \tag{108}$$

commutes.

Another argument showing that the action (104) is the correct one comes from considering bisections of G and the automorphisms of E they generate, according to equation (46). Namely, we can understand this action as the derivative of the push-forward of curves by automorphisms: given a bisection β of G and a curve γ in E, we set $e = \gamma(0)$, $x = \pi_E(e)$ and $g = \beta(x)$ to conclude that if

$$u_g \;=\; T_x\beta \qquad \text{and} \qquad v_e \;=\; \frac{d}{dt}\,\gamma(t)\,\Big|_{t=0}\,, \tag{109}$$

then

$$u_g \cdot v_e \;=\; \frac{d}{dt}\,\Pi_E(\beta)(\gamma(t))\,\Big|_{t=0}\,. \tag{110}$$

Actually, repeating the construction at the end of Section 3.1, we can use the action (104) to obtain a representation of the group $\mathrm{Bis}(JG)$ of bisections of JG by automorphisms of TE (not

only as a fiber bundle over M but also as a vector bundle over E), that is, a homomorphism

$$\begin{array}{rccc} \Pi_{TE}: & \mathrm{Bis}(JG) & \longrightarrow & \mathrm{Aut}(TE) \\ & \tilde{\beta} & \longmapsto & \Pi_{TE}(\tilde{\beta}) \end{array} \tag{111}$$

which covers the homomorphism Π_E defined previously (see equation (46)) in the following sense: if $\tilde{\beta}$ is holonomous, say $\tilde{\beta} = j\beta$, then equations (109) and (110) state that $\Pi_{TE}(\tilde{\beta})$ is the tangent map to $\Pi_E(\beta)$:

$$\Pi_{TE}(j\beta) \;=\; T\,\Pi_E(\beta)\,. \tag{112}$$

The compatibility of the action (104) with the structure of TE as a vector bundle over E allows us to transfer this action to all of its descendants. Thus, for example, we obtain an action of JG on tensors of any degree and type,

$$\begin{array}{rccc} \Phi_{T^r_s E}: & JG \times_M T^r_s E & \longrightarrow & T^r_s E \\ & (u_g, t_e) & \longmapsto & u_g \cdot t_e \end{array} \tag{113}$$

and, in particular, on r-forms,

$$\begin{array}{rccc} \Phi_{\bigwedge^r T^* E}: & JG \times_M \bigwedge^r T^* E & \longrightarrow & \bigwedge^r T^* E \\ & (u_g, \alpha_e) & \longmapsto & u_g \cdot \alpha_e \end{array} \tag{114}$$

that can be restricted to an action of JG on partially horizontal r-forms,

$$\begin{array}{rccc} \Phi_{\bigwedge^r_s T^* E}: & JG \times_M \bigwedge^r_s T^* E & \longrightarrow & \bigwedge^r_s T^* E \\ & (u_g, \alpha_e) & \longmapsto & u_g \cdot \alpha_e \end{array} \tag{115}$$

where $\bigwedge^r_s T^* E$ denotes the bundle of $(r-s)$-horizontal r-forms on E,[3] giving rise to representations of the group of bisections of JG, namely

$$\begin{array}{rccc} \Pi_{T^r_s E}: & \mathrm{Bis}(JG) & \longrightarrow & \mathrm{Aut}(T^r_s E) \\ & \tilde{\beta} & \longmapsto & \Pi_{T^r_s E}(\tilde{\beta}) \end{array} \tag{116}$$

and, in particular,

$$\begin{array}{rccc} \Pi_{\bigwedge^r T^* E}: & \mathrm{Bis}(JG) & \longrightarrow & \mathrm{Aut}(\bigwedge^r T^* E) \\ & \tilde{\beta} & \longmapsto & \Pi_{\bigwedge^r T^* E}(\tilde{\beta}) \end{array} \tag{117}$$

and

$$\begin{array}{rccc} \Pi_{\bigwedge^r_s T^* E}: & \mathrm{Bis}(JG) & \longrightarrow & \mathrm{Aut}(\bigwedge^r_s T^* E) \\ & \tilde{\beta} & \longmapsto & \Pi_{\bigwedge^r_s T^* E}(\tilde{\beta}) \end{array} \tag{118}$$

As a further consistency check, we note the following:

Proposition 3 *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M. Then the canonical strict isomorphism (21) is JG-equivariant.*

The fact that the induced action of JG on TE passes to all its descendants is the key to understanding what is meant by invariance of tensor fields under the action of a Lie groupoid: given a fiber bundle E over a manifold M endowed with an action of a Lie groupoid G over the same manifold M, invariance of a tensor field on E refers to the induced action of the jet groupoid JG on the tangent bundle TE and its descendants. However, in order to give a precise definition, some care must be exerted to deal with holonomous vs. non-holonomous bisections.

Definition 2 *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and let $\tilde{G}$ be a Lie subgroupoid of the jet groupoid JG that is also a subbundle of JG as a fiber bundle over G. Given a tensor field $t \in \mathcal{T}_s^r(E)$ on E, we say that*

*(a) t is **$\tilde{G}$-invariant** if, for all bisections $\tilde{\beta}$ of $\tilde{G}$, $\Pi_{T_s^r E}(\tilde{\beta})_\sharp\, t = t$,*

*(b) t is **G-invariant** if, for all bisections β of G, $\Pi_{T_s^r E}(j\beta)_\sharp\, t = t$,*

where $._\sharp$ denotes the push-forward of sections by automorphisms; note that this coincides with the usual notion $._*$ of push-forward of tensor fields when $\tilde{\beta}$ is holonomous but otherwise provides an extension of that concept, since we have $\Pi_E(\beta)_*\, t = \Pi_{T_s^r E}(j\beta)_\sharp\, t$.

To explain the subtle difference between these two concepts, note that, in general, the group of holonomous bisections of the jet groupoid JG is *not* the group of bisections of any Lie subgroupoid of JG that is also a subbundle of JG as a fiber bundle over G, since the constraint that a bisection of JG should take values in such a Lie subgroupoid involves only its values at points of the base space while the constraint that it should be holonomous involves its first derivatives (which appear through the Frobenius integrability condition). Thus, there is no contradiction between the two parts of Definition 2. But to bring out the difference between them more clearly, it is useful to reformulate them in different terms:
(a) Using the hypothesis that $\tilde{G} \subset JG$, condition (a) of Definition 2 can be reformulated as a pointwise condition: it means that, under the action (113), we have

$$t_{g\cdot e} \;=\; u_g \cdot t_e \tag{119}$$

for all $g \in G$ and $e \in E$ such that $\sigma_G(g) = \pi_E(e)$ and all $u_g \in \tilde{G}_g \subset J_g G$.
(b) According to equation (111), suitably extended from TE to $T_s^r E$, condition (b) of Definition 2 means that t is invariant with respect to push-forward (or pull-back) under the automorphisms $\Pi_E(\beta)$ where β is an arbitrary bisection of G, i.e.,

$$t \text{ is } G\text{-invariant} \quad\Longleftrightarrow\quad \Pi_E(\beta)_*\, t \;=\; t \quad \text{for all } \beta \in \mathrm{Bis}(G)\,. \tag{120}$$

In particular, putting $\tilde{G} = JG$, it becomes obvious that JG-invariance is stronger than G-invariance: somehow, G-invariance can be viewed as a kind of "holonomous JG-invariance".

Let us make these statements even more specific for the case of invariant differential forms. First, the action (114) on r-forms is explicitly determined from the action (104) on tangent vectors as follows: given $g \in G$ and $e \in E$ with $\sigma_G(g) = \pi_E(e)$, $u_g \in J_g G$, $\alpha_e \in \bigwedge^r T_e^* E$ and $v_1, \ldots, v_r \in T_e E$,

$$(u_g \cdot \alpha_e)(u_g \cdot v_1\,, \ldots, u_g \cdot v_r) \;=\; \alpha_e(v_1\,, \ldots, v_r)\,. \tag{121}$$

Thus α will be $\tilde{G}$-invariant iff for any $g \in G$ and $e \in E$ such that $\sigma_G(g) = \pi_E(e)$, any $u_g \in \tilde{G}_g \subset J_g G$ and any tangent vectors $v_1, \ldots, v_r \in T_e E$, we have

$$\alpha_{g \cdot e}(u_g \cdot v_1\,, \ldots, u_g \cdot v_r) \;=\; \alpha_e(v_1\,, \ldots, v_r)\,, \tag{122}$$

while it will be G-invariant iff, for any bisection β of G, we have $\Pi_E(\beta)^* \alpha = \alpha$, or more explicitly, iff for any bisection β of G, any $x \in M$ and $e \in E$ such that $\pi_E(e) = x$ and any tangent vectors $v_1, \ldots, v_r \in T_e E$, we have

$$\alpha_{\beta(x) \cdot e}(T_x\beta \cdot v_1\,, \ldots, T_x\beta \cdot v_r) \;=\; \alpha_e(v_1\,, \ldots, v_r)\,. \tag{123}$$

Note that the second condition is invariant under the action of the exterior derivative: since d commutes witbh pull-back of forms, if α is G-invariant, so is $d\alpha$. This is generally *not true* for the first condition: in fact, $\tilde{G}$-invariance of α will imply $\tilde{G}$-invariance of $d\alpha$ only if $\tilde{G}$ is holonomous, in the sense of Definition 1.

With all these preliminaries out of the way, we are finally ready to show in which sense the multisymplectic structure of classical field theory is invariant:

Theorem 1 *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and consider the induced actions of JG on the extended multiphase space $J^{\circledast}E$ and of the second order jet groupoids $J^2G \subset \bar{J}^2G \subset J(JG)$ on its tangent bundle $T(J^{\circledast}E)$ and its descendants. Then the multicanonical form θ and the multisymplectic form ω are invariant under the action of the second order jet groupoid J^2G.*

In fact, for the multicanonical form θ, we can prove a somewhat stronger statement, namely that it is even invariant under the action of the semiholonomous second order jet groupoid $\bar{J}^2G$. But since this is not holonomous, that invariance does not carry over to its exterior derivative: what remains is invariance under the second order jet groupoid J^2G, which is the maximal holonomous subgroupoid of $\bar{J}^2G$.

Proof: Initially, we remember that the induced actions of JG on JE and on TE cover the original action of G on E (see the diagrams (92) and (107)), and hence the same holds for the induced action of JG on the extended multiphase space of equation (21), which for the sake of brevity we shall here denote by Λ. In other words, the diagram

$$\begin{tikzcd} JG \times_M \Lambda \arrow[r, "\Phi_\Lambda"] \arrow[d, "(\pi_{JG},\pi_\Lambda)"'] & \Lambda \arrow[d, "\pi_\Lambda"] \\ G \times_M E \arrow[r, "\Phi_E"'] & E \end{tikzcd} \tag{124}$$

commutes, that is,

$$\pi_\Lambda \circ \Phi_\Lambda = \Phi_E \circ (\pi_{JG}, \pi_\Lambda).$$

Therefore, given $u_g \in J_g G$ and $\alpha_e \in \Lambda_e$ with $\sigma_G(g) = \pi_E(e)$, $u'_{u_g} \in \bar{J}^2_{u_g} G$ and $w \in T_{\alpha_e}\Lambda$, we

have by equation (105),

$$
\begin{aligned}
T_{u_g\cdot\alpha_e}\pi_\Lambda(u'_{u_g}\cdot w) &= (T_{u_g\cdot\alpha_e}\pi_\Lambda\circ T_{(u_g,\alpha_e)}\Phi_\Lambda)\big((u'_{u_g}\circ T_{\alpha_e}(\pi_E\circ\pi_\Lambda))(w)\,,w\big)\\
&= \big(T_{(g,e)}\Phi_E\circ(T_{u_g}\pi_{JG},T_{\alpha_e}\pi_\Lambda)\big)\big((u'_{u_g}\circ T_{\alpha_e}(\pi_E\circ\pi_\Lambda))(w)\,,w\big)\\
&= T_{(g,e)}\Phi_E\Big(\big(T_{u_g}\pi_{JG}\circ u'_{u_g}\circ T_{\alpha_e}(\pi_E\circ\pi_\Lambda)\big)(w)\,,T_{\alpha_e}\pi_\Lambda(w)\Big)\\
&= T_{(g,e)}\Phi_E\big((u_g\circ T_e\pi_E)(T_{\alpha_e}\pi_\Lambda(w))\,,T_{\alpha_e}\pi_\Lambda(w)\big)\\
&= u_g\cdot(T_{\alpha_e}\pi_\Lambda(w))\,.
\end{aligned}
$$

Thus, given $w_1,\ldots,w_n\in T_{\alpha_e}\Lambda$, we get

$$
\begin{aligned}
&\theta_{u_g\cdot\alpha_e}\big(u'_{u_g}\cdot w_1\,,\ldots,u'_{u_g}\cdot w_n\big)\\
&\quad= (u_g\cdot\alpha_e)\big(T_{u_g\cdot\alpha_e}\pi_\Lambda(u'_{u_g}\cdot w_1)\;\ldots,T_{u_g\cdot\alpha_e}\pi_\Lambda(u'_{u_g}\cdot w_n)\big)\\
&\quad= (u_g\cdot\alpha_e)\big(u_g\cdot(T_{\alpha_e}\pi_\Lambda(w_1))\;\ldots,u_g\cdot(T_{\alpha_e}\pi_\Lambda(w_n))\big)\\
&\quad= \alpha_e\big(T_{\alpha_e}\pi_\Lambda(w_1)\;\ldots,T_{\alpha_e}\pi_\Lambda(w_n)\big)\\
&\quad= \theta_{\alpha_e}\big(w_1\,,\ldots,w_n\big)\,,
\end{aligned}
$$

proving that θ is $\bar{J}^2G$-invariant. That ω is J^2G-invariant then follows from the arguments outlined previously.

The next step is to formulate the concept of invariance of the lagrangian and/or hamiltonian under a Lie groupoid action:

Definition 3 *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and consider the induced actions of JG on the jet bundle JE, the extended multiphase space $J^{\circledast}E$ and the ordinary multiphase space $\vec{J}^{\circledast}E$. Given a Lie subgroupoid $\tilde{G}$ of JG that is also a subbundle of JG as a fiber bundle over G, we say that a lagrangian $\mathcal{L}:JE\longrightarrow\bigwedge^n T^*M$ and/or a hamiltonian $\mathcal{H}:\vec{J}^{\circledast}E\longrightarrow J^{\circledast}E$ is* ***$\tilde{G}$-invariant*** *if it is equivariant with respect to the pertinent actions of JG, when restricted to $\tilde{G}$, i.e., if the diagram*

$$
\begin{tikzcd}
\tilde{G}\times_M JE \arrow[r, "\Phi_{JE}"] \arrow[d, "(\pi^{\mathrm{fr}}_{JG}|_{\tilde{G}},\mathcal{L})"'] & JE \arrow[d, "\mathcal{L}"] \\
GL(TM)\times_M \bigwedge^n T^*M \arrow[r] & \bigwedge^n T^*M
\end{tikzcd}
\tag{125}
$$

and/or the diagram

$$
\begin{tikzcd}
\tilde{G}\times_M \vec{J}^{\circledast}E \arrow[r, "\Phi_{\vec{J}^{\circledast}E}"] \arrow[d, "(\mathrm{id}_{\tilde{G}},\mathcal{H})"'] & \vec{J}^{\circledast}E \arrow[d, "\mathcal{H}"] \\
\tilde{G}\times_M J^{\circledast}E \arrow[r, "\Phi_{J^{\circledast}E}"'] & J^{\circledast}E
\end{tikzcd}
\tag{126}
$$

commutes.

It is noteworthy that $\tilde{G}$-invariance of a lagrangian $\mathcal{L}:JE\longrightarrow\bigwedge^n T^*M$ implies $\tilde{G}$-invariance of its Legendre transformation $\mathbb{F}\mathcal{L}:JE\longrightarrow J^{\circledast}E$, in the sense that the following diagram

commutes:

$$\begin{tikzcd}
\tilde{G} \times_M JE \arrow[r, "\Phi_{JE}"] \arrow[d, "(\mathrm{id}_{\tilde{G}}, \mathbb{F}\mathcal{L})"'] & JE \arrow[d, "\mathbb{F}\mathcal{L}"] \\
\tilde{G} \times_M J^{\circledast}E \arrow[r, "\Phi_{J^{\circledast}E}"'] & J^{\circledast}E
\end{tikzcd} \tag{127}$$

Indeed, given $g \in G$ with $\sigma_G(g) = x$ and $\tau_G(g) = y$, $e \in E$ with $\pi_E(e) = x$, $u_g \in \tilde{G}_g \subset J_g G$ and $u_e, u'_e \in J_e E$, we get, according to equations (28) and (95),

$$\begin{aligned}
&\mathbb{F}\mathcal{L}(u_g \cdot u_e) \cdot (u_g \cdot u'_e) \\
&= \mathcal{L}(u_g \cdot u_e) + \frac{d}{dt}\mathcal{L}\big(u_g \cdot u_e + t(u_g \cdot u'_e - u_g \cdot u_e)\big)\Big|_{t=0} \\
&= \mathcal{L}(u_g \cdot u_e) + \frac{d}{dt}\mathcal{L}\big(u_g \cdot (u_e + t(u'_e - u_e))\big)\Big|_{t=0} \\
&= (T_g\tau_G \circ u_g)^{-1\,*}\mathcal{L}(u_e) + \frac{d}{dt}(T_g\tau_G \circ u_g)^{-1\,*}\mathcal{L}\big(u_e + t(u'_e - u_e)\big)\Big|_{t=0} \\
&= (T_g\tau_G \circ u_g)^{-1\,*}\Big(\mathcal{L}(u_e) + \frac{d}{dt}\mathcal{L}\big(u_e + t(u'_e - u_e)\big)\Big|_{t=0}\Big) \\
&= (T_g\tau_G \circ u_g)^{-1\,*}\big(\mathbb{F}\mathcal{L}(u_e) \cdot u'_e\big) \\
&= (u_g \cdot \mathbb{F}\mathcal{L}(u_e)) \cdot (u_g \cdot u'_e)\,.
\end{aligned}$$

Therefore, invariance of the lagrangian or hamiltonian ensures invariance of the forms $\theta_{\mathcal{L}}$, $\omega_{\mathcal{L}}$ and $\theta_{\mathcal{H}}$, $\omega_{\mathcal{H}}$ defined by pull-back,

$$\theta_{\mathcal{L}} = (\vec{\mathbb{F}}\mathcal{L})^*\theta_{\mathcal{H}} = (\mathbb{F}\mathcal{L})^*\theta \quad , \quad \omega_{\mathcal{L}} = (\vec{\mathbb{F}}\mathcal{L})^*\omega_{\mathcal{H}} = (\mathbb{F}\mathcal{L})^*\omega$$

(see equation (30)), as follows:

Theorem 2 *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and consider the induced actions of JG on the jet bundle JE, the extended multiphase space $J^{\circledast}E$ and the ordinary multiphase space $\vec{J}^{\circledast}E$, as well as of the second order jet groupoids $J^2G \subset \bar{J}^2G \subset J(JG)$ on the respective tangent bundles and their descendants. Given a Lie subgroupoid $\tilde{G}$ of JG that is also a subbundle of JG as a fiber bundle over G, suppose that the lagrangian $\mathcal{L} : JE \longrightarrow \bigwedge^n T^*M$ and/or the hamiltonian $\mathcal{H} : \vec{J}^{\circledast}E \longrightarrow J^{\circledast}E$ are $\tilde{G}$-invariant. Then the forms $\theta_{\mathcal{L}}$, $\omega_{\mathcal{L}}$ and/or $\theta_{\mathcal{H}}$, $\omega_{\mathcal{H}}$ are invariant under $J^2G \cap J\tilde{G}$.*

Proof: This follows directly from Theorem 1 by reformulating the commutativity of the diagram in equation (126) as stating that, for any bisection $\tilde{\beta}$ of $\tilde{G}$,

$$\mathcal{H} \circ \Pi_{\vec{J}^{\circledast}E}(\tilde{\beta}) = \Pi_{J^{\circledast}E}(\tilde{\beta}) \circ \mathcal{H}$$

and hence, for any holonomous bisection $\tilde{\beta}$ of $\tilde{G}$,

$$\Pi_{\vec{J}^{\circledast}E}(\tilde{\beta})^*\big(\mathcal{H}^*\alpha\big) = \mathcal{H}^*\big(\Pi_{J^{\circledast}E}(\tilde{\beta})^*\alpha\big) = \mathcal{H}^*\alpha$$

where α stands for θ or ω. The proof in the lagrangian context is the same, just replacing $\vec{J}^{\circledast}E$ by JE and $\mathcal{H}$ by $\mathbb{F}\mathcal{L}$.

5 Noether's Theorem

We begin this section introducing the concept of momentum map in the context of Lie groupoid actions: it comes in two variants, depending on whether we work in extended or in ordinary multiphase space. To prepare the ground, let us mention that, given a Lie algebroid $\mathfrak{g}$ over M together with a Lie subalgebroid $\tilde{\mathfrak{g}}$ of its jet algebroid $J\mathfrak{g}$, we will define

$$\Gamma(\mathfrak{g},\tilde{\mathfrak{g}}) \;=\; \big\{ X \in \Gamma(\mathfrak{g}) \mid jX \in \Gamma(\tilde{\mathfrak{g}}) \big\}\,, \tag{128}$$

noting that taking the jet prolongation provides a canonical embedding of this space of sections into the space $\Gamma_{\mathrm{hol}}(\tilde{\mathfrak{g}})$ of holonomous sections of $\tilde{\mathfrak{g}}$ that we may use as an identification whenever convenient.

Definition 4 *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and consider the induced actions of JG on the extended multiphase space $J^{\circledast}E$ and the ordinary multiphase space $\vec{J}^{\circledast}E$, as well as the corresponding infinitesimal actions of the Lie algebroids $\mathfrak{g}$ (by fundamental vector fields on E) and $J\mathfrak{g}$ (by fundamental vector fields on $J^{\circledast}E$ and $\vec{J}^{\circledast}E$). Then the* ***extended momentum map*** *$\mathcal{J}^{\mathrm{ext}}$ associated to each of these actions is the map*

$$\mathcal{J}^{\mathrm{ext}} :\; \Gamma(J\mathfrak{g}) \;\longrightarrow\; \Omega^{n-1}\big(J^{\circledast}E\big) \tag{129}$$

defined by

$$\mathcal{J}^{\mathrm{ext}}(Z) \;=\; i_{Z_{J^{\circledast}E}}\theta \tag{130}$$

and the map

$$\mathcal{J}^{\mathrm{ext}} :\; \Gamma(J\mathfrak{g}) \;\longrightarrow\; \Omega^{n-1}\big(\vec{J}^{\circledast}E\big) \tag{131}$$

defined by

$$\mathcal{J}^{\mathrm{ext}}(Z) \;=\; i_{Z_{\vec{J}^{\circledast}E}}\theta_{\mathcal{H}} \tag{132}$$

respectively, and the corresponding ***momentum map*** *is its composition with the jet prolongation map from $\Gamma(\mathfrak{g})$ to $\Gamma(J\mathfrak{g})$, so*

$$\mathcal{J} :\; \Gamma(\mathfrak{g}) \;\longrightarrow\; \Omega^{n-1}\big(J^{\circledast}E\big) \tag{133}$$

with

$$\mathcal{J}(X) \;=\; i_{X_{J^{\circledast}E}}\theta \tag{134}$$

and

$$\mathcal{J} :\; \Gamma(\mathfrak{g}) \;\longrightarrow\; \Omega^{n-1}\big(\vec{J}^{\circledast}E\big) \tag{135}$$

with

$$\mathcal{J}(X) \;=\; i_{X_{\vec{J}^{\circledast}E}}\theta_{\mathcal{H}} \tag{136}$$

where $X_{J^{\circledast}E}$ and $X_{\vec{J}^{\circledast}E}$ are the canonical (dualized jet) lifts of X_E from E to $J^{\circledast}E$ and $\vec{J}^{\circledast}E$, which coincide with the fundamental vector fields $(jX)_{J^{\circledast}E}$ and $(jX)_{\vec{J}^{\circledast}E}$, respectively.

Only the ordinary multiphase space version appears directly in Noether's theorem:

Theorem 3 (Noether's theorem) *Let E be a fiber bundle over a manifold M, endowed with the action of a Lie groupoid G over the same manifold M, and consider the induced action of JG on the ordinary multiphase space $\vec{J}^{\circledast}E$, as well as the corresponding infinitesimal actions of the Lie algebroids $\mathfrak{g}$ (by fundamental vector fields on E) and $J\mathfrak{g}$ (by fundamental vector fields on $\vec{J}^{\circledast}E$). Given a Lie subgroupoid $\tilde{G}$ of JG that is also a subbundle of JG as a fiber bundle over G, with corresponding Lie subalgebroid $\tilde{\mathfrak{g}}$ of $J\mathfrak{g}$, and a $\tilde{G}$-invariant hamiltonian $\mathcal{H} : \vec{J}^{\circledast}E \longrightarrow J^{\circledast}E$, the* ***Noether current*** *associated with a "generator" $X \in \Gamma(\mathfrak{g}, \tilde{\mathfrak{g}})$ and a section ϕ of $\vec{J}^{\circledast}E$ is the pull-back $\phi^* \mathcal{J}(X) \in \Omega^{n-1}(M)$. Then if ϕ satisfies the equations of motion, i.e., the De Donder – Weyl equations, this current is* ***conserved****, i.e., a closed form:*

$$d[\phi^* \mathcal{J}(X)] \;=\; 0\,.$$

Proof: Given $X \in \Gamma(\mathfrak{g}, \tilde{\mathfrak{g}})$ and $\phi \in \Gamma(\vec{J}^{\circledast}E)$, we have

$$\begin{aligned} d[\phi^* \mathcal{J}(X)] \;&=\; d[\phi^*(i_{X_{\vec{J}^{\circledast}E}} \theta_{\mathcal{H}})] \;=\; \phi^* d(i_{X_{\vec{J}^{\circledast}E}} \theta_{\mathcal{H}}) \\ &=\; \phi^*(L_{X_{\vec{J}^{\circledast}E}} \theta_{\mathcal{H}}) \,+\, \phi^*(i_{X_{\vec{J}^{\circledast}E}} \omega_{\mathcal{H}})\,. \end{aligned}$$

Claim 1:

$$L_{X_{\vec{J}^{\circledast}E}} \theta_{\mathcal{H}} = 0.$$

By hypothesis, X generates a one-parameter subgroup of bisections $\exp(tX)$ of G such that the prolonged bisections $j(\exp(tX))$ of JG take values in $\tilde{G}$, and $X_{\vec{J}^{\circledast}E}$ generates the one-parameter subgroup $\Pi_{\vec{J}^{\circledast}E}(j(\exp(tX)))$ of automorphisms of $\vec{J}^{\circledast}E$. But since $\mathcal{H}$ is $\tilde{G}$-invariant and hence, according to Theorem 2, $\theta_{\mathcal{H}}$ is invariant under $J^2G \cap J\tilde{G}$, we can just apply Definition 2 (a), with the substitutions $E \to \vec{J}^{\circledast}E$, $t \to \theta_{\mathcal{H}}$, $G \to JG$, $\tilde{G} \to J^2G \cap J\tilde{G}$ and $\tilde{\beta} \to j^2(\exp(tX))$, to conclude that

$$\Pi_{\vec{J}^{\circledast}E}(j(\exp(tX))_* \, \theta_{\mathcal{H}} \;=\; \Pi_{\bigwedge^n T(\vec{J}^{\circledast}E)}(j^2(\exp(tX))_\sharp \, \theta_{\mathcal{H}} \;=\; \theta_{\mathcal{H}}$$

and hence

$$L_{X_{\vec{J}^{\circledast}E}} \theta_{\mathcal{H}} \;=\; \frac{d}{dt} \, \Pi_{\vec{J}^{\circledast}E}(j(\exp(tX)))^* \, \theta_{\mathcal{H}} \Big|_{t=0} \;=\; \frac{d}{dt} \, \theta_{\mathcal{H}} \Big|_{t=0} \;=\; 0\,.$$

Claim 2: If ϕ is a solution of the De Donder – Weyl equations, then since $X_{\vec{J}^{\circledast}E}$ is projectable, it follows from equation (31) that

$$\phi^*(i_{X_{\vec{J}^{\circledast}E}} \omega_{\mathcal{H}}) \;=\; 0\,.$$

□

6 Conclusions and Outlook

In this paper, we have taken first steps towards a description of symmetries in field theory using Lie groupoids and Lie algebroids, instead of the traditional approach that uses Lie groups and Lie algebras but requires infinite-dimensional ones as soon as local symmetries are involved. Our main motivation for doing so arises from the observation that in relativistic field theories, the need to consider local symmetries is almost unavoidable, since here the notion of a global

symmetry – which, by definition, applies the same transformation at every point of space-time – is a mathematical artifact without physical meaning. After all, any physical implementation of such a requirement of rigidity would violate the principle of space-time locality, according to which no information can be exchanged between space-like separated regions of space-time. (This is really the same argument as the one showing that in relativistic mechanics there is no such thing as a rigid body.) We argue that the theory of Lie groupoids and Lie algebroids provides the adequate mathematical machinery to describe local symmetries in field theory; in particular, this applies to gauge theories, whose geometric formulation using principal bundles and connections has become standard wisdom during the 1970's. In this context, it may be amusing to note that Ehresmann already invented all the essential mathematical notions (principal bundles, connections *and* Lie groupoids) during the 1950's, more or less in one stroke, but mathematicians have for several decades used only the first two and largely neglected the third, and this lack of balance has proliferated into physics.[8] So in order to incorporate Lie groupoids and Lie algebroids into the picture, we have to catch up on four decades of delay.

As an example of what can be gained, we may quote the classical difficulties with unravelling the true symmetry of typical lagrangians in field theory over curved space-times, provided we are interested in including space-time symmetries. In the traditional group-theoretical approach, the pertinent symmetry is given by the isometry group of space-time (in special relativity, the Poincaré group), which may collapse to a trivial group under arbitrarily small perturbations of the metric. Much of this instability disappears when we use groupoids: what appears there is the orthonormal frame groupoid of space-time, which is quite stable under arbitrary perturbations (even large ones), but is generically non-holonomous. Thus the notion of holonomous or non-holonomous subgroupoids of jet groupoids, which has no analogue in traditional group theory, appears to be an essential tool for understanding this issue. We plan to elaborate further on this point in the second part of this series.

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References

[1] R. Abraham & J.E. Marsden: *Foundations of Mechanics*, 2nd edition, Benjamin-Cummings, Reading 1978.

[2] V.I. Arnold: *Mathematical Methods of Classical Mechanics*, 3rd edition, Springer, Berlin 1978.

[3] H. Brandt: *Über eine Verallgemeinerung des Gruppenbegriffs*, Math. Ann. **96** (1927) 360-366.

[4] J.F. Cariñena, M. Crampin & L.A. Ibort: *On the multisymplectic formalism for first order field theories*, Diff. Geom. Appl. **1** (1991) 345-374.

[5] M. Crainic & R.L. Fernandes: *Integrability of Lie brackets*, Ann. Math. **157** (2003) 575-620.

[8] As a testimony to this statement, we may quote the classical textbooks of Kobayahi and Nomizu.

[6] M. Forger & B.L. Soares: *Local symmetries in gauge theories in a finite-dimensional setting*, J. Geom. Phys. **62** (2012) 1925-1938.

[7] M. Forger & S.V. Romero: *Covariant Poisson brackets in geometric field theory*, Commun. Math. Phys. 256 (2005) 375-410.

[8] M.J. Gotay, J. Isenberg, J.E. Marsden & R. Montgomery: *Momentum maps and classical relativistic fields I: covariant field theory*, `arXiv:physics/9801019`.

[9] I. Hargittai (ed.): *Symmetry: Unifying Human Understanding*, Pergamon Press, Elmsford 1986 & Computers and Mathematics with Applications **12 B** (1986) Issues 1-4.

[10] I. Hargittai (ed.): *Symmetry 2: Unifying Human Understanding*, Pergamon Press, Elmsford 1989 & Computers and Mathematics with Applications **17** (1989) Issues 1-6.

[11] K.C.H. Mackenzie: *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, Cambridge 2005.

[12] J. Pradines: *Troisième théorème de Lie sur les groupoïdes différentiables*, C. R. Acad. Sci. Paris Sér. A **267** (1968) 21-23.

[13] D.J. Saunders: *The Geometry of Jet Bundles*, Cambridge University Press, Cambridge 1989.